A Drift-Kinetic Analytical Model for SOL Plasma Dynamics at Arbitrary Collisionality

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A drift-kinetic model to describe the plasma dynamics in the scrape-off layer region of tokamak devices at arbitrary collisionality is derived. Our formulation is based on a gyroaveraged Lagrangian description of the charged particle motion, and the corresponding drift-kinetic Boltzmann equation that includes a full Coulomb collision operator. Using a Hermite-Laguerre velocity space decomposition of the gyroaveraged distribution function, a set of equations to evolve the coefficients of the expansion is presented. By evaluating explicitly the moments of the Coulomb collision operator, distribution functions arbitrarily far from equilibrium can be studied at arbitrary collisionalities. A fluid closure in the high-collisionality limit is presented, and the corresponding fluid equations are compared with previously-derived fluid models.

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1. Introduction

The success of the magnetic confinement fusion program relies on our ability to predict the dynamics of the plasma in the tokamak scrape-off layer (SOL). In this region, the plasma is turbulent with fluctuation level of order unity (Ritz \textit{et al.} 1987; Wootton \textit{et al.} 1987).

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The fluctuations are characterized by frequencies lower than the ion gyrofrequency (Endler et al. 1995; Agostini et al. 2011; Carralero et al. 2014; Garcia et al. 2015), and the turbulent eddies, which include coherent radial propagation of filamentary structures (D’Ippolito et al. 2002, 2011; Carreras 2005; Serianni et al. 2007), have a radial extension comparable to the time-averaged SOL pressure gradient length $L_p$ (Zweben et al. 2007).

In recent years, there has been a significant development of first-principles simulations of the SOL dynamics with both kinetic (Tskhakaya 2012) and gyrokinetic (Xu et al. 2007; Shi et al. 2015; Chang et al. 2017; Shi et al. 2017) codes. However, as kinetic simulations of the SOL and edge regions remain prohibitively as they still are computationally extremely expensive, the less demanding fluid simulations are the standard of reference (Dudson et al. 2009; Tamain et al. 2009; Easy et al. 2014; Halpern et al. 2016; Madsen et al. 2016). The fluid simulations are usually based on the drift-reduced Braginskii (Braginskii 1965; Zeiler et al. 1997) or gyrofluid (Ribeiro & Scott 2008; Held et al. 2016) models to evolve plasma density, fluid velocity and temperature. Fluid models assume that the distribution function is close to a local Maxwellian, and that scale lengths along the magnetic field are longer than the mean free path. However, kinetic simulations show that the plasma distribution function is far from Maxwellian in the SOL region (Tskhakaya et al. 2008; Lönroth et al. 2006; Battaglia et al. 2014) and that collisionless effects in the SOL might become important (Battishchev et al. 1997). This is expected to be particularly true in ITER and other future devices that will be operated in the high confinement mode (H-mode) regime (Martin et al. 2008). In such cases, a transport barrier is formed that creates a steep pressure gradient at the plasma edge. If the pressure gradient exceeds a threshold value, edge-localized modes (ELMs) are destabilized (Leonard 2014), expelling large amounts of heat and particles to the wall. Describing structures with such high temperatures (and therefore low-collisionality) with respect to the background SOL plasma requires therefore a model that allows for the treatment of arbitrary collision frequencies. Higher moments of the distribution function are needed for a proper SOL description (Hazeltine 1998).

Leveraging the development of previous models (Hammett et al. 1993; Beer & Hammett 1996; Ji & Held 2010; Zocco & Schekochihin 2011; Schekochihin et al. 2016; Hatch et al. 2016; Parker 2016; Hirvijoki et al. 2016; Mandell et al. 2017), we construct here a moment hierarchy to evolve the SOL plasma dynamics. Our model is valid in arbitrary magnetic field geometries and, making use of the full Coulomb collision operator, at arbitrary collision frequencies. The model is derived within a full-F framework, as the amplitude of the background and fluctuating components of the plasma parameters in the SOL have comparable amplitude. We work within the drift approximation (Hinton & Hazeltine 1976; Cary & Brizard 2009), which assumes that plasma quantities have typical frequencies that are small compared to the ion gyrofrequency $\Omega_i = eB/m_i$, and their perpendicular spatial scale is small compared to the ion sound Larmor radius $\rho_s = c_s/\Omega_i$, with $c_s^2 = T_e/m_i$, $T_e$ the electron temperature, and $m_i$ the ion mass. Moreover, we consider a Braginskii ordering (Braginskii 1965), where the species flow velocities are comparable to the ion thermal speed, as opposed to the drift ordering introduced by Mikhailovskii & Tsypin (1971) and extended and corrected by Catto & Simakov (2004), where flow velocities are comparable to the diamagnetic drift velocities.

More precisely, denoting $k_\perp \sim |\nabla_\perp \log \phi| \sim |\nabla_\perp \log n| \sim |\nabla_\perp \log T_e|$ and $\omega \sim |\partial_t \log \phi| \sim |\partial_t \log n| \sim |\partial_t \log T_e|$, with $\phi$ the electrostatic potential and $n$ the plasma density, we introduce the ordering parameter $\epsilon$ such that

$$\epsilon \sim k_\perp \rho_s \sim \omega/\Omega_i \ll 1.$$  (1.1)
On the other hand, we let $k_\perp L_p \sim 1$ since turbulent eddies are observed to have an extension comparable to the scale lengths of the time-averaged quantities. These assumptions are in agreement with experimental measurements of SOL plasmas (LaBombard et al. 2001; Zweben et al. 2004; Myra et al. 2013; Carralero et al. 2014). We also order the electron collision frequency $\nu_e$ as

$$\frac{\nu_e}{\Omega_i} \sim \epsilon, \quad (1.2)$$

In addition, the ion collision frequency $\nu_i = \nu_{ii}$ is ordered as $\nu_{ii} \lesssim \epsilon^2 \Omega_i$ that, noticing $\nu_i \sim \sqrt{m_e/m_i(T_e/T_i)^{3/2}}\nu_e$ (with $\nu_e = \nu_{ei}$), yields

$$\left(\frac{\epsilon \nu}{\epsilon^2} \right)^{2/3} \left(\frac{m_e}{m_i}\right)^{1/3} \lesssim \frac{T_i}{T_e} \lesssim 1. \quad (1.3)$$

The ordering in Eq. (1.3) can be used to justify applying our model in the cold ion limit, $T_i \ll T_e$, but allows for $T_i \sim T_e$. We note that in the SOL the ratio $T_i/T_e$ is in the range $1 \lesssim T_i/T_e \lesssim 4$ (Kočan et al. 2011). The ion temperature in this range of values is seen to play a negligible role in determining the SOL turbulent dynamics, usually due to a steeper electron temperature profile compared with the ion one, which is usually below the threshold limit of the ion temperature gradient instability (Mosetto et al. 2015).

The ordering in Eqs. (1.1)-(1.3) is justified in a wide variety of experimental conditions. For example, for a typical JET discharge (Erents et al. 2000; Liang et al. 2007; Xu et al. 2009) with the SOL parameters $B_T = 2.5$ T, $T_e \sim T_i \sim 20$ eV, $n_e \simeq 10^{19}$ m$^{-3}$, and $k_\perp \sim 1$ cm$^{-1}$, we obtain $\epsilon \nu \sim 0.016$ and $\epsilon \sim 0.0182$. For a medium-size tokamak such as TCV (Rossel et al. 2012; Nespoli et al. 2017), estimating $B_T = 1.5$ T, $T_e \sim T_i \sim 40$ eV, $n_e \simeq 6 \times 10^{18}$, and $k_\perp \sim 1$ cm$^{-1}$, we obtain $\epsilon \nu \sim 6.2 \times 10^{-3}$ and $\epsilon \sim 0.043$. Finally, for small-size tokamaks such as ISTTOK (Silva et al. 2011; Jorge et al. 2016), with $B_T = 0.5$ T, $T_e \sim T_i \sim 20$ eV, $n_e \simeq 0.8 \times 10^{18}$, and $k_\perp \sim 1$ cm$^{-1}$, we obtain $\epsilon \nu \sim 0.0072$ and $\epsilon \sim 0.091$. Lower values of $\epsilon \nu$, as in the presence of ELMs where temperatures can reach up to 100 eV (Pitts et al. 2003), are also included in the ordering considered here.

Following typical SOL experimental measurements (see, e.g. Zweben et al. (2007); Terry et al. (2009); Grulke et al. (2014)), we order $k_\parallel \sim 1/L_B \sim 1/R$, with $L_B$ the background magnetic field spatial gradient scale and $R$ the tokamak major radius, and take $k_\parallel \rho_s \sim \epsilon^3$. This yields

$$k_\parallel \sim \epsilon^2, \quad (1.4)$$

a lower ratio than the ones used in most drift-kinetic and gyrokinetic deductions (Hahm 1988; Hazeltine & Meiss 2003; Abel et al. 2013). The orderings in Eqs. (1.2) and (1.4) imply that

$$k_\parallel \lambda_{mfp} \sim \sqrt{\frac{m_i \epsilon^3}{m_e \epsilon \nu}}, \quad (1.5)$$

which includes both the collisional regime $k_\parallel \lambda_{mfp} \lesssim 1$, when $\epsilon \nu \sim \epsilon$, and the collisionless regime $k_\parallel \lambda_{mfp} \gg 1$, when $\epsilon \nu \ll \epsilon$. Finally, the plasma parameter $\beta = nT_e/(B^2/2\mu_0)$ is ordered as $\beta \lesssim \epsilon^3$, implying that our equations describe plasma dynamics in an electrostatic regime.

Our model describes the evolution of the moments of the drift-kinetic Boltzmann equation at order $\epsilon^2$, taking into account the effect of collisions through a full Coulomb collision operator. The kinetic equation is based on a Lagrangian description of the charged
particle motion. In the SOL, due to the large fluctuations and the short characteristic gradient width \( L_\phi \sim L_p \), a strong electric field is present. To properly retain the effect of a non-negligible \( \mathbf{E} \times \mathbf{B} \) drift, \( \mathbf{v}_E = -\nabla \phi \times \mathbf{B}/B^2 \), in the equations of motion, we split the perpendicular component of the particle velocity \( \mathbf{v}_\perp \) into \( \mathbf{v}_\perp = \mathbf{v}_E + \mathbf{v}_\perp' \). In the particle Lagrangian, we keep the resulting term \( m \mathbf{v}_E^2/2 \) associated with the \( \mathbf{E} \times \mathbf{B} \) motion of the gyrocenters, as it will be shown to be of the same order of magnitude as the first-order terms in the Lagrangian (see Krommes (2013) for a discussion on the physical interpretation of this term).

In the kinetic equation, we expand the gyroaveraged distribution function into a Hermite-Laguerre basis, and express the moments of the collision operator in a series of products of the expansion coefficients of the distribution function. For like-species collisions, this expansion is based on the work of Ji & Held (2009), while we make use of the small mass ratio approximation to obtain electron-ion and ion-electron operators that ensure basic conservation properties. The system is closed by Poisson’s equation, involving explicitly the moments of the distribution function, accurate up to order \( \epsilon^2 \) (we also present a derivation of Poisson’s equation that rigorously includes collisional \( \epsilon \nu \) effects).

This paper is organized as follows. Section 2 derives the equations of motion of a charged particle in the SOL, and the drift-kinetic Boltzmann equation in a conservative form. In Section 3 we expand the gyroaveraged distribution function in a Hermite-Laguerre basis and obtain the guiding-center moments of the collision operator. In Section 4 we take moments of the drift-kinetic Boltzmann equation, and deduce the moment-hierarchy equations. Section 5 presents the guiding-center Poisson’s equation, accurate up to order \( \epsilon^2 \). Finally, in Section 6, a fluid model based on the truncation of the Hermite-Laguerre expansion in the high-collisionality regime is presented. The conclusions follow. Appendix A presents the transformation between pitch-angle and parallel-perpendicular velocity basis. Appendix B lists explicitly the moments of the parallel acceleration phase-space conserving term. Appendix C derives Poisson’s equation with higher-order collisional effects. Finally, in Appendix D, the the lower order guiding-center moments of the collision operator are given in explicit form.

2. SOL Guiding-Center Model

2.1. Single Particle Motion

To derive a convenient equation of motion in the presence of a strong magnetic field \( \mathbf{B} \), we start with the Hamiltonian of a charged particle of species \( a \) (Jackson 1998),

\[
H_a(x, p) = \frac{[p - q_a \mathbf{A}]^2}{2m_a} + q_a \phi, \tag{2.1}
\]

and its associated Lagrangian,

\[
L_a(x, v) = [q_a \mathbf{A}(x) + m_a \mathbf{v}] \cdot \dot{x} - \left( \frac{m_a v^2}{2} + q_a \phi(x) \right), \tag{2.2}
\]

where \( p = m_a \mathbf{v} + q_a \mathbf{A} \), \( \mathbf{v} \) is the particle velocity, \( \mathbf{A} \) is the magnetic vector potential, \( \phi \) is the electrostatic potential, \( m_a \) is the mass of the particle and \( q_a \) its charge.

We now perform a coordinate transformation from the phase-space coordinates \( z = (x, v) \) to the guiding-center coordinates \( Z = (R, v_\parallel, \mu, \theta) \) by writing the particle velocity as (see, e.g., Littlejohn (1983))
\[ v = U + v'_c, \]  

with \( U = v_E(R) + v \parallel b(R), \) \( v \parallel = v \cdot b, b = B/B, \) and \( v_E = E \times B/B^2. \) The gyroangle \( \theta = \tan^{-1}[(v - U) \cdot e_2/\langle (v - U) \cdot e_1 \rangle] \) is introduced by defining the right-handed coordinate set \( (e_1, e_2, b), \) such that \( c = -a \times b = a'(\theta), \) with \( a = \cos \theta e_1 + \sin \theta e_2. \) The decomposition in Eq. (2.3) allows us to isolate the high-frequency gyromotion, contained in the \( v'_c \) term, from the dominant guiding-center velocity \( U. \) The adiabatic moment \( \mu \) is defined as

\[ \mu = \frac{m_a v'_c^2}{2B}, \]  

whereas the guiding-center position is

\[ R = x - \rho_a a, \]  

with \( \rho_a = \sqrt{2m_a \mu/(q_a^2 B)} \) the Larmor radius. Incidentally, for the case of weakly varying magnetic fields, Eq. (2.5) describes the circular motion of a particle around its guiding-center \( R \) with radius \( \rho_a, \) i.e., \( (x - R)^2 = \rho_a^2. \)

As our goal is to develop a model that describes turbulent fluctuations occurring on a spatial scale longer than the sound Larmor radius \( \rho_s, \) and a time scale larger than the gyromotion one, we keep terms in the Lagrangian up to \( O(\epsilon) \) and order \( T_i \lesssim T_e, \) which implies

\[ k_\perp \rho_i \lesssim \epsilon. \]  

We therefore expand the electromagnetic fields around \( R, \) to first order in \( \epsilon, \) i.e.,

\[ \phi(x) \simeq \phi(R) + \rho_a a \cdot \nabla_R \phi(R), \]  

and similarly for \( A. \) In the following, if not specified, the electromagnetic fields and potentials are evaluated at the guiding-center position \( R, \) and we denote \( \nabla = \nabla_R. \) In addition, to take advantage of the difference between the turbulent and gyromotion time scales, we use the gyroaveraged Lagrangian \( \langle L_a \rangle \) to evaluate the plasma particle motion, where the gyroaveraging operator \( \langle \chi \rangle \) acting on a quantity \( \chi(\theta) \) is defined as

\[ \langle \chi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \chi(\theta)d\theta, \]

which is performed at fixed position \( x, \) as opposed to the gyrokinetic equation that can be obtained by gyroaveraging with \( R \) fixed (Hazeltine & Meiss 2003).

To evaluate \( \langle L_a \rangle \) we note that, with the expansion for \( \phi \) and \( A, \) the Lagrangian in Eq. (2.2) can be expressed as \( L_a = L_{0a} + L_{1a} + \tilde{L}_a \) where \( L_{0a} \) is gyroangle independent,

\[ L_{0a} = (q_a A + m_a U) \cdot \dot{R} - \left( \frac{m_a v^2_\parallel}{2} + \frac{m_a v^2_E}{2} + \mu B + q_a \phi \right), \]

\( L_{1a} \) is proportional to \( \rho_a^2 \) (and hence to \( \mu),

\[ L_{1a} = \rho_a^2 q_a \dot{\theta} (A \cdot \nabla) (A \cdot c) + m_a \rho_a^2 \Omega \dot{\theta} + \rho_a \dot{\rho}_a [q_a (A \cdot \nabla) (A \cdot a)], \]

and the \( \tilde{L}_a \) contribution contains the terms linearly proportional to \( \cos \theta \) or \( \sin \theta \) (Cary & Brizard 2009) which are not present in \( \langle L_a \rangle \) as \( \langle \tilde{L}_a \rangle = 0. \)
We note that $\langle L_{1a} \rangle$ can be simplified since $\langle (a \cdot \nabla) \mathbf{A} \cdot c \rangle = -b \cdot (\nabla \times \mathbf{A})/2$, and $\langle (a \cdot \nabla) \mathbf{A} \cdot a \rangle = \nabla_\perp \cdot \mathbf{A}/2$. Subtracting the total derivative $-q_a d/dt(\rho_a^2 \nabla_\perp \mathbf{A})/4$ from $\langle L_a \rangle$, which does not alter the resulting equations of motion, we redefine the gyroaveraged Lagrangian as

$$
\langle L_a \rangle = (q_a \mathbf{A} + m_a \mathbf{U}) \cdot \dot{\mathbf{R}} - \left( \frac{m_a v_\parallel^2}{2} + \frac{m_a v_E^2}{2} + q_a \phi \right)
- \mu B \left( 1 - \frac{\dot{\theta}}{\Omega_a} \right) - \frac{\rho_a^2}{4} \frac{d}{dt} [\nabla_\perp \cdot (q_a \mathbf{A})].
$$

(2.11)

We now order the terms appearing in $\langle L_a \rangle$. As imposed by the Bohm sheath conditions (Stangeby 2000), both electrons and ions stream along the field lines with parallel velocities comparable to the sound speed $c_s = \sqrt{T_e/m_i}$ in the SOL. The Bohm boundary conditions at the sheath also set the electrostatic potential $e\phi \sim AT_e$ across the SOL, where $A = \log \sqrt{m_i/(m_e 2\pi)} \simeq 3$. Therefore, we keep the $m_a v_E^2/2$ term in the Lagrangian in Eq. (2.11), as to take into account the presence of the numerically large factor $A^2$ in $v_E^2 \sim e^2 A^2 c_s^2$.

By neglecting the higher-order terms in Eq. (2.11), i.e., $-(\rho_a^2/4) [\nabla_\perp \cdot (q(a \mathbf{A})] / dt$, the expression for the gyroaveraged Lagrangian describing SOL single particle dynamics, up to $O(\epsilon)$, can be written as

$$
\langle L_a \rangle = q_a \mathbf{A}^* \cdot \dot{\mathbf{R}} - q_a \phi^* - \frac{m_a v_\parallel^2}{2} + \frac{m_a \dot{\phi}}{q_a},
$$

(2.12)

where $q_a \phi^* = q_a \phi + m_a v_\parallel^2/2 + \mu B$, and $q_a \mathbf{A}^* = q_a \mathbf{A} + m_a v_\parallel \mathbf{b} + m_a v_E \mathbf{b}$. The Euler-Lagrange equations applied to the Lagrangian in Eq. (2.12) for the coordinates $\theta$, $v_\parallel$, and $\mu$, yield, respectively, $\epsilon = 0$, $v_\parallel = \mathbf{b} \cdot \dot{\mathbf{R}}$, and $\dot{\theta} = \Omega_a$. For the $R$ coordinate, we obtain

$$
m_a v_\parallel b = q_a (E^* + \dot{R} \times B^*),
$$

(2.13)

where the relation $(\nabla \mathbf{A} - (\nabla \mathbf{A})^T) \cdot \dot{\mathbf{R}} = \dot{\mathbf{R}} \times (\nabla \times \mathbf{A})$ has been used, and we defined $E^* = -\nabla \phi^* - \partial_t \mathbf{A}^*$, and $B^* = \nabla \times \mathbf{A}^*$, with the parallel component of $B^*$ given by

$$
B_\parallel^* = B^* \cdot \mathbf{b} = B + \frac{m_a}{q_a} \mathbf{b} \cdot \nabla \times (v_\parallel \mathbf{b} + v_E).
$$

(2.14)

By projecting Eq. (2.13) along $B^*$, we derive $m \dot{v}_\parallel B^*_\parallel = e E^* \cdot B^*$, while crossing with $\mathbf{b}$ yields the guiding-center velocity $\dot{R} B^*_\parallel = v_\parallel B^* + E^* \times B/B$. Using the expressions for the fields $E^*$ and $B^*$, we obtain

$$
\dot{\mathbf{R}} = \mathbf{U} + \frac{\mathbf{B}}{\Omega_a B^*_\parallel} \times \left( \frac{d\mathbf{U}}{dt} + \frac{\mu \nabla B}{m_a} \right),
$$

(2.15)

and

$$
m_a \dot{v}_\parallel = q_a E_\parallel - \mu \nabla_B + m_a v_E \cdot \frac{db}{dt} - m_a A,
$$

(2.16)

In Eqs. (2.15) and (2.16), in addition to the time derivatives of the phase-space coordinates $\dot{\mathbf{R}}, \dot{v}_\parallel$, that only have an explicit time dependence, we define the total derivative $d/dt$ of a field $\phi(\mathbf{R}, t)$ that has an explicit time and $\mathbf{R}$ dependence as $d\phi/dt \equiv \partial_t \phi + \mathbf{U} \cdot \nabla \phi$. 

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The $\mathcal{A}$ term represents the higher-order nonlinear terms in $\dot{v}_\parallel$ that ensure phase-space conservation properties (Cary & Brizard 2009), and it is given by

$$\mathcal{A} = \frac{B}{B_\parallel} \left( \frac{dU}{dt} \right)_{\perp} + \mu \nabla_{\perp} B \cdot \nabla \times U, \quad (2.17)$$

The guiding-center equations of motion (2.15) and (2.16) satisfy the energy, $E_{gc} = q_a \phi^* + m_a v_\parallel^2 / 2$ (Cary & Brizard 2009), and momentum, $P_{gc} = e A^*$ (Cary & Brizard 2009), conservation laws, i.e.,

$$\frac{dE_{gc}}{dt} = q_a \frac{\partial \phi^*}{\partial t} - q_a \frac{\partial A^*}{\partial t} \cdot \dot{R}, \quad (2.18)$$

and

$$\frac{\partial P_{gc}}{\partial t} = -q_a \nabla \phi^* + q_a \nabla A^* \cdot \dot{R}. \quad (2.19)$$

In addition, we note that using Eqs. (2.15) and (2.16) and Maxwell’s equations, a conservation equation for $B^*_{\parallel}$ can be derived:

$$\frac{\partial B^*_{\parallel}}{\partial t} + \nabla \cdot (\dot{R} B^*_\parallel) + \frac{\partial}{\partial v_{\parallel}} \left( \dot{v}_{\parallel} B^*_\parallel \right) = 0. \quad (2.20)$$

2.2. The Guiding-Center Boltzmann Equation

The Boltzmann equation for the evolution of the distribution function $f_a(x, v)$ of the particles in $(x, v)$ coordinates is

$$\frac{\partial f_a}{\partial t} + \dot{x} \cdot \nabla_x f_a + \dot{v} \cdot \nabla_v f_a = C(f_a), \quad (2.21)$$

where $C(f_a) = \sum_b C(f_a, f_b) = \sum_b C_{ab}$ is the collision operator. Because $f_a$ can significantly deviate from a Maxwellian distribution function in the SOL (Battaglia et al. 2014), we consider the bilinear Coulomb operator $C_{ab}$ (Balescu 1988), to model collisions between particles of species $a$ and $b$

$$C_{ab} = L_{ab} \frac{\partial}{\partial v_i} \left[ \frac{\partial^2 G_b}{\partial v_i \partial v_j} \frac{\partial f_a}{\partial v_j} - \frac{m_a}{m_b} \frac{\partial H_b}{\partial v_i} f_a \right]. \quad (2.22)$$

with

$$H_b = 2 \int \frac{f_b(v')}{|v - v'|} dv', \quad (2.23)$$

and

$$G_b = \int f_b(v') |v - v'| dv', \quad (2.24)$$

the Rosenbluth potentials satisfying $\nabla^2 v G_b = H_b$. In Eq. (2.22) we introduced $L_{ab} = q_a^2 q_b^2 \lambda / (4\pi e_0^2 m_a^2) = \nu_{ab} v_{tha}^2 / n_b$, where $\lambda$ is the Coulomb logarithm, $\nu_{ab}$ the collision frequency between species $a$ and $b$, and $v_{tha}^2 = 2T_a / m_a$.

Taking advantage of the small electron to ion mass ratio, the collision operator between
Unlike-species can be simplified (see, e.g. Balescu (1988); Helander & Sigmar (2005)). The electron-ion collisions are modeled by using the operator 

$$C_{ei}^0 = \frac{n_i L_{ei}}{8\pi} \frac{\partial}{\partial c_e} \left[ \frac{1}{c_e} \frac{\partial f_e}{\partial c_e} - \frac{c_e}{c_e^3} \left( c_e \cdot \frac{\partial f_e}{\partial c_e} \right) \right],$$

(2.25)

and $C_{ei}^1$ the momentum-conserving term

$$C_{ei}^1 = \frac{n_i L_{ei}}{4\pi c_e^2} f_{Me} u_{ei} \cdot c_e.$$

(2.26)

with $c_e = (v - u_e)/v_{the}$ and $c_{ei} = (u_e - u_i)/v_{the}$ the normalized difference between the electron $u_e$ and ion $u_i$ fluid velocities.

Ion-electron collisions are modeled with the operator

$$C_{ie} = \frac{R_{ei}}{m_i n_i v_{thi}} \frac{\partial f_i}{\partial c_i} + \nu_{ei} \frac{n_e m_e}{n_i m_i} \frac{\partial}{\partial c_i} \left( c_i f_i + \frac{T_e}{T_i} \frac{\partial f_i}{\partial c_i} \right),$$

(2.27)

where $R_{ei} = \int m_e v C_{ei} dv$ is the electron-ion friction force.

We take advantage of Eq. (1.2) to order the electron collision frequency $\nu_e$ and the ion collision frequency $\nu_i$ as

$$\frac{\nu_i}{\Omega_i} \sim \sqrt{\frac{m_e}{m_i} \left( \frac{T_e}{T_i} \right)^{3/2}} \epsilon_\nu \lesssim \epsilon^2,$$

(2.28)

where we used the relation $\nu_i \sim \sqrt{m_e/m_i (T_e/T_i)^{3/2}} \nu_e$. The orderings in Eqs. (2.7) and (2.28) yield the lower bound in Eq. (1.3) for the ion to electron temperature ratio. We now express the particle distribution function $f_a$ in terms of the guiding-center coordinates by defining $F_a$, a function of guiding-center coordinates, as

$$F_a(R, v_\parallel, \mu, \theta) = f_a(x(R, v_\parallel, \mu, \theta), v(R, v_\parallel, \mu, \theta)).$$

(2.29)

Using the chain rule to rewrite Eq. (2.21) in guiding-center coordinates, we obtain

$$\frac{\partial F_a}{\partial t} + \hat{R} \cdot \nabla F_a + v_\parallel \frac{\partial F_a}{\partial v_\parallel} + \hat{\mu} \frac{\partial F_a}{\partial \mu} + \hat{\theta} \frac{\partial F_a}{\partial \theta} = C(F_a),$$

(2.30)

where $\hat{R}$ and $v_\parallel$ are given by Eq. (2.15) and Eq. (2.16) respectively, $\hat{\theta} = \Omega_a$, and $\hat{\mu} = 0$. Equation (2.30) can be simplified by applying the gyroaveraging operator. This results in the drift-kinetic equation

$$\frac{\partial \langle F_a \rangle}{\partial t} + \hat{R} \cdot \nabla \langle F_a \rangle + v_\parallel \frac{\partial \langle F_a \rangle}{\partial v_\parallel} = \langle C(F_a) \rangle.$$

(2.31)

We now write Eq. (2.31) in a form useful to take gyrofluid moments of the form $\int \langle F_a \rangle B_{a\parallel} d\mu d\theta$ (see Section 4). Using the conservation law in Eq. (2.20) for $B_\parallel^a$, we can write the guiding-center Boltzmann equation in conservative form as

$$\frac{\partial (B_\parallel^a \langle F_a \rangle)}{\partial t} + \nabla \cdot (\hat{R} B_\parallel^a \langle F_a \rangle) + \frac{\partial \langle v_\parallel B_\parallel^a \langle F_a \rangle \rangle}{\partial v_\parallel} = B_\parallel^a \langle C(F_a) \rangle.$$

(2.32)

Moreover, in order to relate the gyrofluid moments $\int \langle F_a \rangle B_{a\parallel} d\mu d\theta$ with the usual fluid moments $\int f_a d^3v$, we estimate the order of magnitude of the gyrophase dependent part of
the distribution function \( \tilde{F}_a = F_a - \langle F_a \rangle \) where \( \langle F_a \rangle \) obeys Eq. (2.31). The equation for the evolution of \( \tilde{F}_a \) is obtained by subtracting Eq. (2.31) from the Boltzmann equation, Eq. (2.30), that is

\[
\frac{\partial \tilde{F}_a}{\partial t} + \tilde{\mathbf{R}} \cdot \nabla \tilde{F}_a + \tilde{v}_i \frac{\partial \tilde{F}_a}{\partial v_i} + \Omega_a \frac{\partial \tilde{F}_a}{\partial \theta} = C(F_a) - \langle C(F_a) \rangle .
\] (2.33)

Using the orderings in Eqs. (1.2) and (2.28), as well as \( \partial_t \sim \dot{\mathbf{R}} \cdot \nabla \sim \dot{v}_i \frac{\partial}{\partial v_i} \sim \epsilon \Omega_i \) and \( \Omega_a \partial \theta \sim \Omega_a \), the comparison of the leading-order term on the left-hand side of Eq. (2.33) with the right-hand side of the same equation imply the following ordering for \( \tilde{F}_e \)

\[
\frac{\tilde{F}_e}{\langle F_e \rangle} \sim \frac{m_e}{m_i} \epsilon \nu \lesssim \epsilon^2,
\] (2.34)

and \( \tilde{F}_i \)

\[
\frac{\tilde{F}_i}{\langle F_i \rangle} \sim \sqrt{\frac{m_e}{m_i}} \left( \frac{T_e}{T_i} \right)^{3/2} \epsilon \nu \lesssim \epsilon^2.
\] (2.35)

To evaluate the leading-order term of \( \tilde{F}_a \), we expand the collision operator \( C(F_a) = C_0(\langle F_a \rangle) + \epsilon C_1(F_a) + \ldots \), such that

\[
\tilde{F}_a \approx \frac{1}{\Omega_a} \int_0^\theta \left[ C_0(\langle F_a \rangle) - \langle C_0(\langle F_a \rangle) \rangle \right] d\theta' + O(\epsilon^3 \langle F_a \rangle),
\] (2.36)

The relation in Eq. (2.36) can be further simplified by expanding the \( \theta \) dependence of \( F_a \) in Fourier harmonics,

\[
F_a = \sum_m e^{im\theta} F_{ma},
\] (2.37)

so that for \( m = 0 \) we have \( \langle F_a \rangle = F_{0a} \), and similarly for \( C_0(\langle F_a \rangle) \)

\[
C_0(\langle F_a \rangle) = \sum_{m'} e^{im'\theta} C_{m'a}.
\] (2.38)

We can then write Eq. (2.36) as

\[
\tilde{F}_{ma} = \frac{C_{ma}}{im\Omega_a},
\] (2.39)

for \( m \neq 0 \).

### 3. Moment Expansion

We now derive a polynomial expansion for the distribution function \( \langle F_a \rangle \) that simplifies the solution of Eq. (2.32), with the collision operators in Eqs. (2.22) - (2.27). This section is organized as follows. In Section 3.1 the Hermite-Laguerre basis is introduced, relating the corresponding expansion coefficients for \( \langle F_a \rangle \) with its usual gyrofluid moments. In Section 3.2, we briefly review the fluid moment expansion of the Coulomb collision operator presented in Ji & Held (2006, 2008). In Section 3.3, leveraging the work in Ji & Held (2006, 2008), we expand \( C_{ab} \) in terms of the product of the gyrofluid moments, for both like- and unlike-species collisions. This ultimately gives us the possibility of solving Eq. (2.32) in terms of gyrofluid moments.
3.1. Guiding-Center Moment Expansion of $\langle F_a \rangle$

To take advantage of the anisotropy introduced by a strong magnetic field, and efficiently treat the left-hand side of Eq. (2.32) where the parallel and perpendicular directions appear decoupled, we express $\langle F_a \rangle$ by using a Hermite polynomial basis expansion for the parallel velocity coordinate (Grad 1949; Armstrong 1967; Grant & Feix 1967; Ng et al. 1999; Zocco & Schekochihin 2011; Loureiro et al. 2013; Parker & Dellar 2015; Schekochihin et al. 2016; Tassi 2016) and a Laguerre polynomial basis for the perpendicular velocity coordinate (Omotani et al. 2015; Mandell et al. 2017). More precisely, we use the following expansion

$$\langle F_a \rangle = \sum_{p,j=0}^{\infty} \frac{N^p_j}{\sqrt{2^p p!}} F_{Ma} H_p(s_{\parallel a}) L_j(s_{\perp a}^2),$$

(3.1)

where the physicists’ Hermite polynomials $H_p$ of order $p$ are defined by (Abramowicz et al. 1988)

$$H_p(x) = (-1)^p e^{-x^2} \frac{d^p}{dx^p} e^{-x^2},$$

(3.2)

and normalized via

$$\int_{-\infty}^{\infty} dx H_p(x) H_{p'}(x) e^{-x^2} = 2^p p! \sqrt{\pi} \delta_{pp'},$$

(3.3)

and the Laguerre polynomials $L_j$ of order $j$ are defined by (Abramowicz et al. 1988)

$$L_j(x) = e^x \frac{d^j}{dx^j} (e^{-x} x^j),$$

(3.4)

which are orthonormal with respect to the weight $e^{-x}$

$$\int_0^{\infty} dx L_j(x) L_{j'}(x) e^{-x} = \delta_{jj'}.$$  

(3.5)

Because of the orthogonality of the Hermite-Laguerre basis, the coefficients $N^p_j$ of the expansion in Eq. (3.1) are

$$N^p_j = \frac{1}{N_a} \int H_p(s_{\parallel a}) L_j(s_{\perp a}^2) \langle F_a \rangle \frac{B}{m_a} d\mu dv_{\parallel} d\theta,$$

(3.6)

and correspond to the guiding-center moments of $\langle F_a \rangle$.

In Eq. (3.1), the shifted bi-Maxwellian is introduced

$$F_{Ma} = N_a \frac{e^{-s_{\parallel a}^2 - s_{\perp a}^2}}{\pi^{3/2} v_{th\parallel a} v_{th\perp a}^2},$$

(3.7)

where $s_{\parallel a}$ and $s_{\perp a}$ are the normalized parallel and perpendicular shifted velocities respectively, defined by

$$s_{\parallel a} = \frac{v_{\parallel} - u_{\parallel a}}{v_{th\parallel a}}, \quad s_{\perp a}^2 = \frac{2T_{\parallel a}}{m_a},$$

(3.8)

and
The fluid variables \( f \) which provide an efficient representation of the distribution function in both the weak (\( u_{||a} \ll v_{tha} \)) and strong flow (\( u_{||a} \sim v_{tha} \)) regimes.

The guiding-center density \( N_a \), appearing in Eq. (3.7), the guiding-center fluid velocity \( u_{||a} \), in Eq. (3.8), and the guiding-center parallel \( T_{||a} = P_{||a}/N_a \) and perpendicular \( T_{\perp a} = P_{\perp a}/N_a \) temperatures in Eqs. (3.8) and (3.9) are defined as \( N_a = ||1||_a, N_a u_{||a} = ||v||_a, P_{||a} = m_a ||(v - u_{||a})^2||_a, \) and \( P_{\perp a} = ||\mu B||_a, \) where

\[
||\chi||_a \equiv \int \chi \langle F_a \rangle \frac{B}{m_a} d\mu dv d\theta. \tag{3.10}
\]

The definition of \( N_a, u_{||a}, P_{||a}, \) and \( P_{\perp a} \) implies that \( N_a^{00} = 1, N_a^{10} = 0, N_a^{20} = 0, N_a^{01} = 0 \) respectively. Later, we will consider the parallel and perpendicular heat flux, defined as

\[
Q_{||a} = m_a ||(v - u_{||a})^3||_a, \quad Q_{\perp a} = ||(v - u_{||a})\mu B||_a, \tag{3.11}
\]

which are related to the coefficients \( N_a^{30}, N_a^{11} \) by

\[
N_a^{30} = \frac{Q_{||a}}{\sqrt{3}P_{||a} v_{tha}||}, \quad N_a^{11} = -\frac{\sqrt{2}Q_{\perp a}}{P_{\perp a} v_{tha}||}. \tag{3.12}
\]

### 3.2. Fluid Moment Expansion of the Collision Operator

A polynomial expansion of the collision operators in Eq. (2.22) was carried out in Ji & Held (2006), and later extended to effectively take into account finite fluid velocity and unlike-species collisions in Ji & Held (2008). This allowed expressing \( C_{ab} \) as products of fluid moments of \( f_a \) and \( f_b \). We summarize here the main steps of Ji & Held (2006, 2008).

Similarly to Eq. (3.1), the particle distribution function \( f_a \) is expanded as

\[
f_a = f_{aM} \sum_{l,k=0}^{\infty} \frac{L_k^{l+1/2}(c_a^2) P^l(c_a) \cdot M_a^{lk}}{\sqrt{\sigma_k^l}}, \tag{3.13}
\]

where \( f_{aM} = n_a \exp(-c_a^2)/(\pi^{3/2}v_{tha}^3) \) is a shifted Maxwell-Boltzmann distribution function, and \( c_a \) the shifted velocity defined as \( c_a = (v - U_a)/v_{tha} \), with \( U_a \) the fluid velocity. The fluid variables \( n_a, U_a, \) and \( T_a \) are defined as the usual moments of the particle distribution function \( f_a \), i.e. \( n_a = \int f_a d^3v, n_a U_a = \int f_a v d^3v, n_a T_a = \int m f_a (v - u_a)^2 d^3v/3. \)

The tensors \( P_a^{lk}(c_a) = P^l(c_a) L_k^{l+1/2}(c_a^2) \) constitute an orthogonal basis, where \( P^l(c_a) \) is the symmetric and traceless tensor

\[
P^l(c_a) = \sum_{i=0}^{[l/2]} d_i^l S_i^l c_{a2i}^l \{ l^i c_{a}^{l-2i} \}, \tag{3.14}
\]

with \( I \) denoting the identity matrix, \( \{ A^i \} \) denoting the symmetrization of the tensor \( A^i \), \([l/2]\) denoting the largest integer less than or equal to \( l/2 \), and the coefficients \( d_i^l \) and \( S_i^l \) defined by

\[
S_i^l = \frac{1}{2} \sum_{m=-l}^l \binom{l}{m} \frac{1}{m+1} \frac{\Gamma(l+1)}{\Gamma(m+1)\Gamma(l-m+1)}, \quad d_i^l = (-1)^{l-i} \binom{l-i}{l}.
\]
\[ d_l^i = \frac{(-2)^i(2l - 2i)!!}{(2l)!(l - i)!} , \]  
\[ S_l^i = \frac{l!}{(l - 2i)!!i!} . \]

The tensor \( P^l(c_a) \) is normalized via

\[
\int dv P^n(v) P^l(v) \cdot M^l g(v) = M^n \delta_{n,l} \sigma_n \int dv v^{2n} g(v) ,
\]

with \( \sigma_l = l!/[(2l + 1/2)!] \). We note that the tensor \( A^i \) is formed by \( i \) multiplications of the \( A \) elements (e.g., if \( A \) is a rank-2 tensor, \( A^3 = AAA \), which in index notation can be written as \( (A^3)_{ijkmn} = A_{ij}A_{ik}A_{mn} \)).

In the expansion in Eq. (3.13), \( L_{k}^{l+1/2}(x) \) are the associated Laguerre polynomials

\[ L_{k}^{l+1/2}(x) = \sum_{m=0}^{k} L_{km} x^m , \]  
normalized via

\[
\int_{0}^{\infty} e^{-x} x^{l+1/2} L_{k}^{l+1/2}(x) L_{k'}^{l+1/2}(x) dx = \lambda_k^l \delta_{k,k'} .
\]

Finally, the coefficients of the expansion in Eq. (3.13) \( M_{a}^{lk} \) are

\[ M_{a}^{lk} = \frac{1}{n_a} \int dv f_a L_{k}^{l+1/2}(c_a^2) P^l(c_a) , \]  
which correspond to the moments of \( f_a \) due to the orthogonality relations in Eqs. (3.17) and (3.19).

By using the expansion in Eq. (3.13) in the collision operator in Eq. (2.22), a closed form for \( C_{ab} \) in terms of products of \( M_{a}^{lk} \) can be obtained. For like-species collisions it reads

\[ C_{aa} = \sum_{lk} \sum_{nm} \frac{L_{km}^l L_{nr}^n}{\sqrt{\sigma_k^l \sigma_q^n}} c(f_{a}^{lk}, f_{a}^{nq}) , \]  
with

\[
c(f_{a}^{lk}, f_{a}^{nq}) = f_{aM} \sum_{u=0}^{\min(2,l,n)} \nu_{saa}^{lmnr} (c_a^2) \sum_{i=0}^{\min(l,n)-u} d_{i}^{l-u,n-u} P^{l+n-2(i+u)}(c_a) \cdot M_{a}^{lk+i+u} M_{a}^{nq} ,
\]

where \( \cdot^n \) is the \( n \)-fold inner product (e.g., for the matrix \( A = A_{ij} \), \( (A \cdot A)_{ij} = \))
∑_k A_{ki} A_{kj}), and \( \overline{A} \) the traceless symmetrization of \( A \) (e.g., \( \overline{A} = (A_{ij} + A_{ji})/2 - \delta_{ij} \sum_k A_{kk}/3 \)). We refer the reader to Ji & Held (2009) for the explicit form of the \( \nu^{lm, nr}_{sub} \) coefficients. The expansion of the unlike-species collisions is reported in Ji & Held (2008).

3.3. Guiding-Center Moment Expansion of the Collision Operators

In order to apply the gyroaveraging operator to the like-species collision operator \( C_{aa} \) in Eq. (3.21), we expand the fluid moments as \( M_{lk}^{a0} = M_{lk}^{a0} + \epsilon M_{lk}^{a1} + ... \), aiming at representing the collision operator up to \( O(\epsilon, \epsilon) \). An analytical expression for the leading-order \( M_{lk}^{a0} \) in terms of guiding-center moments \( N_{pj}^{a} \) can be obtained as follows. By splitting \( f_{a} = \langle f_{a} \rangle + \tilde{f}_{a} \) when evaluating the fluid moments \( M_{lk}^{a} \) according to Eq. (3.20), we obtain

\[
M_{lk}^{a} = \frac{1}{n_{a}} \int d^3 x' d^3 v' \delta(x' - x) \frac{I_{k}^{l} + \epsilon^{l}}{\sigma_{k}^{l}} P_{l}^{\prime}(c_{a}^{\prime}) \left( \langle f_{a} \rangle + \tilde{f}_{a} \right).
\] (3.23)

where the Dirac delta function was introduced to convert the velocity integral into an \((x, v)\) integral that encompasses the full phase-space. Since the volume element in phase space can be written as \( d^3 x d^3 v = (B_{*}/m) dR dv dµ dθ \) (Cary & Brizard 2009), and defining \( x' = R + \rho_{a} a \), we can write the fluid moments in Eq. (3.23) as

\[
M_{lk}^{a} = \frac{1}{n_{a}} \int dR dv dµ dθ \frac{B_{*}}{m_{a}} \delta(x - R - \rho_{a} a) \frac{I_{k}^{l} + \epsilon^{l}}{\sigma_{k}^{l}} P_{l}^{\prime}(c_{a}^{\prime}) \left( \langle f_{a} \rangle + \tilde{f}_{a} \right).
\] (3.24)

where \( \langle f_{a} \rangle \) and \( \tilde{f}_{a} \) in Eq. (3.23) are written in terms of guiding-center coordinates using Eq. (2.29). Neglecting the higher-order \( \rho_{a} \) and \( \tilde{F}_{a} \) terms, the leading-order fluid moments \( M_{lk}^{a0} \) are given by

\[
M_{lk}^{a0} = \frac{1}{n_{a}} \int dv dµ dθ \frac{B_{*}}{m_{a}} \frac{I_{k}^{l} + \epsilon^{l}}{\sigma_{k}^{l}} P_{l}^{\prime}(c_{a}^{\prime}) \langle f_{a} \rangle.
\] (3.25)

The \( \theta \) integration can be performed by making use of the gyroaveraging formula of the \( P^{l} \) tensor (Ji & Held 2009)

\[
\langle P^{l}(c_{a}) \rangle = c_{a}^{l} P_{l}(\xi_{a}) \ P^{l}(b),
\] (3.26)

where \( \xi_{a} = c_{a} \cdot b/c_{a} \) is the pitch angle velocity coordinate, and \( P_{l} \) is a Legendre polynomial defined by

\[
P_{l}(x) = \frac{1}{2^{l} l!} \frac{d^{l}}{dx^{l}} [(x^{2} - 1)^{l}],
\] (3.27)

and normalized via

\[
\int_{-1}^{1} P_{l}(x) P_{l'}(x) dx = \frac{\delta_{ll'}}{l + 1/2},
\] (3.28)

yielding
\[ M_{a0}^{lk} = \frac{P^l(b)}{n_a} \int d\nu d\mu d\theta \frac{B^*}{m_a} \frac{L_k^{l+1/2}(c_a^2)P_l(\xi_a)}{\sqrt{\sigma_k^l}} \langle F_a \rangle. \]  

(3.29)

Finally, we use the basis transformation

\[ c_a^l P_l(\xi_a)L_k^{l+1/2}(c_a^2) = \sum_{p=0}^{l+2k-1/2} \sum_{j=0}^{k} T_{a(lk)}^{pj} H_p(s_{\parallel a})L_j(s_{\perp a}^2), \]  

(3.30)

with the inverse

\[ H_p(s_{\parallel a})L_j(s_{\perp a}^2) = \sum_{l=0}^{p+2j-1} \sum_{k=0}^{j+\lfloor p/2 \rfloor} (T_a^{-1})_{pkj}^l c_a^l P_l(\xi_a)L_k^{l+1/2}(c_a^2), \]  

(3.31)

to obtain an expression in terms of the Hermite-Laguerre basis. A numerical evaluation of \( T_{a(lk)}^{pj} \) and \( (T_a^{-1})_{pkj}^l \) was carried out in Omotani et al. (2015). Instead, in Appendix A, we derive the analytic expressions of both \( T_{a(lk)}^{pj} \) and \( (T_a^{-1})_{pkj}^l \).

Using the definition of guiding-center moments \( N_{a}^{pj} \) in Eq. (3.6), the leading-order fluid moment \( M_{a0}^{lk} \) is then given by

\[ n_a M_{a0}^{lk} = N_a P^l(b) N_{a}^{lk}, \]  

(3.32)

where we define

\[ N_{a}^{lk} = \sum_{p=0}^{l+2k-1/2} \sum_{j=0}^{k} T_{a(lk)}^{pj} N_{a}^{pj} \sqrt{\frac{2p!}{\sigma_k^l}}. \]  

(3.33)

The leading-order part \( C_{a0a} \) of the collision operator \( C_{aa} \) can be calculated by approximating \( M_{a0}^{lk} \) appearing in Eq. (3.22) with \( M_{a0}^{lk} \). For the ions, the largest contribution to \( M_{i0}^{lk} - M_{i0}^{lk} \) is of order \( \epsilon \) and it is given by the \( \rho_i \) appearing in Eq. (3.24) (the \( \tilde{F}_i \) correction is smaller since \( \tilde{F}_i \lesssim \epsilon^2 \langle F_i \rangle \), see Eq. (2.35)). Therefore, by using the ordering in Eq. (2.28), the largest correction to \( C_{ii0} \) is \( O(\sqrt{m_e/m_i \epsilon \nu}) \). The correction to \( C_{e0a} \) is of the same order. It follows that we can approximate \( C_{aa} \) appearing in Eq. (3.22) with \( C_{aa0} \) to represent the collision operator up to \( O(\epsilon \nu \epsilon) \).

As an aside, we note that the relationship between the guiding-center and fluid moments in Eq. (3.32) provides, for the indices \((l,k) = (0,0)\),

\[ n_a = N_a, \]  

(3.34)

while, for \((l,k) = (0,1)\),

\[ T_a = \frac{T_{\parallel a} + 2T_{\perp a}}{3}. \]  

(3.35)

Moreover, the \((l,k) = (2,0)\) moment provides a relationship useful to express the viscosity tensor \( \Pi_a = \int (c_a c_a - c_a^2 I) f_a d\nu \)

\[ \Pi_a = bb N(T_{\parallel a} - T_{\perp a}), \]  

(3.36)
and \((l, k) = (1, 1)\) gives

\[
q_a = \left( \frac{Q_{\|a}}{2} + Q_{\perp a} \right) b, \tag{3.37}
\]

with \(q_a\) the heat flux density \(q_a = m \int c_a c_a^2 f_a dv/2\).

In order to express the Boltzmann equation, Eq. (2.32), in terms of the guiding-center moments \(N^q_a\), we evaluate the guiding-center moments of \(\langle C_{aa} \rangle\), namely

\[
C^{pq}_{aa} = \frac{1}{N_a} \int \langle C_{aa0} \rangle \frac{H_p(s_{\|a}) L_j(s_{\perp a}^2)}{\sqrt{2p!}} B_{m_a} dv d\mu d\theta. \tag{3.38}
\]

By using the gyroaveraging property (3.26) of \(P^i(c_a)\) in the like-species operator in Eqs. (3.21) and (3.22) (with \(M_a^{lk} = M_a^{ik}\)), and the relation (3.32) between \(M_a^{lk}\) and \(N^q_a\), the gyroaveraged collision operator coefficients \(\langle c (f_a^{lkm}, f_a^{nqr}) \rangle\) are given by

\[
\langle c (f_a^{lkm}, f_a^{nqr}) \rangle = f_a M_a \sum_{u=0}^{\min(2, l, n)} \nu_{saa}^{l,m,n} (c_a^2) \sum_{i=0}^{\min(l,n)-u} d_i^{l-u,n-u} P_{l+n-2(i+u)}(\xi) N_k^l N^q_a P_{i+u}^{l,n}, \tag{3.39}
\]

with \(P_{i+u}^{l,n} = P_{l+n-2(i+u)}^{l+1/2} \cdot P_{i+u}^{l,n}\).

Using the basis transformation of Eq. (3.31) to express \(H_p(s_{\|a}) L_j(s_{\perp a}^2)\) in Eq. (3.38) in terms of \(c_a^l P_l(\xi) L^{l+1/2}_k (c_a^2)\), and performing the resulting integral, we obtain

\[
C_{aa}^{pq} = \sum_{l, k} \sum_{n, q} \sum_{u=0}^{\min(2, l, n)} \sum_{i=0}^{\min(l,n)-u} \sum_{e=0}^{p+2j} \sum_{f=0}^{|p/2|} \frac{L_k^{l-1} L_j^{n-1} e^{d_i^{l-u,n-u}}}{\sqrt{\sigma_k^l \sigma_q^n}} \frac{C_{sa}^{l,m,n}}{\sqrt{2p!}} \delta_{e,l+n-2(i+u)} (T^{-1})^{ef}_{p} N_k^l N^q_a P_{i+u}^{l,n}, \tag{3.40}
\]

with \(C_{sa}^{l,m,n} = \int dv c_a^{2u+j} f_{Ma}^{l,m,n}\).

We now turn to the electron-ion collision operator, \(C_{ei} = C_{ei}^0 + C_{ei}^1\), with \(C_{ei}^0\) given by Eq. (2.25) and \(C_{ei}^1\) given by Eq. (2.26). As the basis \(L_k^{l+1/2} P^i(c_e)\) is an eigenfunction of the Lorentz pitch-angle scattering operator \(C_{ei}^0\) with eigenvalue \(-l(l+1)\) (Ji & Held 2008), we write \(C_{ei}^0\) as

\[
C_{ei}^0 = -\sum_{l, k} \frac{n_k L_k^{l+1} f_{eM} f_k^{l+1/2} (c_e^2) P^i(c_e) \cdot M_e^{lk}}{8 \pi c_e^2 \sqrt{\sigma_k^l}}. \tag{3.41}
\]

Similarly to like-species collisions, we approximate \(M_e^{lk} \simeq M_e^{lk}_0\) in Eq. (3.41), representing \(C_{ei}^0\) accurately up to \(O(\nu_e, \epsilon)\). Using the basis transformation in Eq. (3.31) and the gyroaverage property of \(P^i(c_a)\) in Eq. (3.26), we take guiding-center moments of \(C_{ei}\) of the form (3.38), and obtain

\[
C_{ei}^{pq} = -\frac{\nu_e}{8 \pi c_e^2} \sum_{l, k} \sum_{e=0}^{p+2j} \sum_{f=0}^{|p/2|} \frac{(T_e^{-1})^{ef}_{p} \delta_{e, l+n-2(i+u)} (T^{-1})^{ef}_{p} N_k^l N^q_a P_{i+u}^{l,n}}{\sqrt{2p!}} \sum_{k=0}^{\infty} A_{e}^{lf,k} N_{e}^{lk} - \delta_{l+1} u_{\|i} - u_{\|e} 8 \Gamma (f + 3/2) \frac{3}{f! \sqrt{\pi}}, \tag{3.42}
\]
where the $A_{ei}$ coefficients are given by

$$A_{ei,0}^{l,f,k} = \frac{l(l+1)}{l+1/2} |P^l(b)|^2 \sum_{m=0}^{f} \sum_{n=0}^{k} \frac{L_m^l L_n^k}{\sqrt{\sigma_k^{l}}} (l + m + n + 1)! \quad (3.43)$$

Finally, for the ion-electron collision operator, $C_{ie}$, we neglect $O(\sqrt{m_e/m_\perp \epsilon_{\perp}})$ corrections by approximating $F_i \simeq \langle F_i \rangle$, and use the transformation in Eq. (2.3) to convert the $C_{ie}$ operator in Eq. (2.27) to guiding-center variables, yielding

$$C_{ie} = \frac{R_{ei}}{m_i n_i v_{thi}} \left[ c_{1} \frac{m_i v_{thi}^2}{B} \frac{\partial \langle F_i \rangle}{\partial \mu} + b \frac{\partial \langle F_i \rangle}{\partial c_{||i}} \right] + \nu_{ei} \frac{m_e n_e}{m_i n_i} \left[ 3 \langle F_i \rangle + c_{1} \frac{\partial \langle F_i \rangle}{\partial c_{||i}} + 2 \mu \frac{\partial \langle F_i \rangle}{\partial \mu} + \frac{T_e}{2T_i} \frac{\partial^2 \langle F_i \rangle}{\partial c_{||i}^2} + \frac{2T_e}{B} \frac{\partial}{\partial \mu} \left( \mu \frac{\partial \langle F_i \rangle}{\partial \mu} \right) \right]. \quad (3.44)$$

By evaluating $R_{ei}$ at the guiding-center position $R$, we write $R_{ei} \cdot b = N_{e} m_e v_{thi} C_{ei}^{10} / \sqrt{2} + O(\sqrt{m_e/m_\perp \epsilon_{\perp}})$ and gyroaverage Eq. (3.44), yielding

$$\langle C_{ie} \rangle = \frac{C_{ei}^{10}}{\sqrt{2}} \frac{m_e n_e}{m_i n_i} \frac{v_{thi}}{v_{thi}^{||}} \frac{\partial \langle F_i \rangle}{\partial s_{||}} + \nu_{ei} \frac{m_e n_e}{m_i n_i} \left[ 3 \langle F_i \rangle + s_{||} \frac{\partial \langle F_i \rangle}{\partial s_{||}} + 2 \mu \frac{\partial \langle F_i \rangle}{\partial \mu} + \frac{T_e}{2T_i} \frac{\partial^2 \langle F_i \rangle}{\partial s_{||}^2} + \frac{2T_e}{B} \frac{\partial}{\partial \mu} \left( \mu \frac{\partial \langle F_i \rangle}{\partial \mu} \right) \right], \quad (3.45)$$

where we used $c_{1}^{2} = s_{||}^{2} T_{||}/T_{i}$. Taking guiding-center moments of the form (3.38) of $\langle C_{ie} \rangle$ in Eq. (3.45), we obtain

$$C_{ie}^{pq} = \nu_{ei} \frac{m_e}{m_i} \sum_{lk} B_{lk}^{pq} N_{i}^{lk}, \quad (3.46)$$

with

$$B_{lk}^{pq} = 2j \delta_{lk} \delta_{kj-1} \left( 1 - \frac{T_e}{T_{i}} \right) - \sqrt{p} \frac{v_{thi}}{v_{thi}^{||}} \frac{C_{ei}^{10}}{\nu_{ei}} \delta_{lk-1} \delta_{kj}$$

$$- (p + 2j) \delta_{lk} \delta_{kj} + \sqrt{p(p-1)} \delta_{lk} \delta_{kj} \left( \frac{T_e}{T_{i}} - 1 \right). \quad (3.47)$$

### 4. Moment Hierarchy

In this section, we derive a set of equations that describe the evolution of the guiding-center moments $N_{i}^{pq}$, by integrating in guiding-center velocity space the conservative form of the Boltzmann equation, Eq. (2.32), with the weights $H_{p}(s_{||}) L_{j}(s_{\perp}^{2})$. First, we highlight the dependence of $\dot{R}$ and $\dot{v}_{||}$ on $s_{||}$ and $s_{\perp}^{2}$ by rewriting the equations of motion as

$$\dot{R} = U_{0a} + U_{pa}^* + s_{\perp}^{2} U_{p \perp a}^* + s_{||}^{2} U_{k a}^* + s_{||} (v_{thi})_{a} b + U_{pa}^{*th}, \quad (4.1)$$

and
\[ m_a \dot{v}_\parallel = F_{\parallel a} - s_{\perp a}^2 F_{Ma} + s_{\parallel a} F_{pa}^{th} - m_a A. \] (4.2)

In Eqs. (4.1) and (4.2), \( U_{0a} = v_E + u_{\parallel a} b \) is the lowest-order guiding-center fluid velocity, \( U_{\nabla B a} = (T_{\perp a} / m_a) (b \times \nabla B / \Omega_a^* B) \) is the fluid grad-B drift, with \( \Omega_a^* = q_a B_{\parallel} / m_a \), \( U_{ka} = (2 T_{\parallel a} / m_a) (b \times k / \Omega_a^*) \) is the fluid curvature drift with \( k = b \cdot \nabla b \), \( U_{*pa} = (b / \Omega_a^*) \times d_0 U_{0a} / dt \) is the fluid polarization drift, \( F_{\parallel a} = q_a E_{\parallel} + m_a v_{E} \cdot d_0 b / dt \), \( F_{Ma} = T_{\perp a} \nabla \ln B \) is the mirror force, and both \( U_{*pa} \) and \( F_{*pa} \) are related to gradients of the electromagnetic fields

\[ U_{*pa} = v_{th\parallel a} \frac{b}{\Omega_a^*} \times (b \cdot \nabla v_E + v_E \cdot \nabla b + 2 u_{\parallel a} k), \] (4.3)

\[ F_{*pa} = m_a v_{th\parallel a} b \cdot \left( \frac{k \times E}{B} \right). \]

The fluid convective derivative operator is defined as

\[ \frac{d_{0a}}{dt} = \partial_t + U_{0a} \cdot \nabla. \] (4.4)

Next, to obtain an equation for the moment \( N_{pj}^a \), we apply the guiding-center moment operator

\[ ||\chi||_a^{*pj} = \frac{1}{N_a B} ||\chi H_p(s_{\parallel a}) L_j(s_{\perp a}^2) B_{\parallel||}^*|| \]

\[ = \frac{1}{N_a} \int \frac{B_{\parallel||}^*}{m_a} \langle F_a \rangle \frac{H_p(s_{\parallel a}) L_j(s_{\perp a}^2)}{\sqrt{2p}} dv_{\parallel} d\mu d\theta, \] (4.5)

where \( \langle F_a \rangle \) is the flux of the electromagnetic fields to Boltzmann’s equation, Eq. (2.32). By defining \( ||1||_a^{*pj} = N_{pj}^a \) such that

\[ N_{pj}^a = N_{pj}^a \left( 1 + \frac{b \cdot \nabla \times v_E}{\Omega_a} + u_{\parallel a} \frac{b \cdot \nabla \times b}{\Omega_a} \right) \]

\[ + v_{th\parallel a} \frac{b \cdot \nabla \times b}{\sqrt{2} \Omega_a} \left( \sqrt{p + 1} N_{pj}^{p+1} + \sqrt{p} N_{pj}^{p-1} \right), \] (4.6)

and

\[ \frac{d_{pj}^{*pj}}{dt} = N_{pj}^a \frac{\partial}{\partial t} + \frac{\partial \langle R \rangle_a^{*pj}}{\partial t} \cdot \nabla, \] (4.7)

the drift-kinetic moment hierarchy conservation equation for species \( a \) is

\[ \frac{\partial N_{pj}^a}{\partial t} + \nabla \cdot \left| \langle R \rangle_a^{*pj} \right| - \frac{\sqrt{2p}}{v_{th\parallel a}} \left| \dot{v}_{\parallel} \right| \left| \langle R \rangle_a^{*pj} \right| - F_{pj}^a = \sum_b C_{ab}^{pj}, \] (4.8)

where we identify the fluid operator...
The expressions of where the phase-mixing operators read velocity polynomials in the Hermite-Laguerre basis, which gives rise to the fluid operator hierarchy equation, Eq. (4.8). In kinetic moments fluid operator projection of the Coulomb operator in velocity space. We also note that the use of shifted that couple nearby Hermite and Laguerre moments and providing a close form for the spatially varying fields and full Coulomb collisions, while retaining phase-mixing operators region of validity (Camporeale even shown to be more efficient than other velocity discretization techniques in the same De Ninno 2009; Loureiro implemented, and successfully compared with their kinetic counterpart (Paškauskas & et al. 2016; Schekochihin et al. 2016; Grošelj et al. 2017), and even shown to be more efficient than other velocity discretization techniques in the same region of validity (Camporeale et al. 2016). Equation (4.8) generalizes such models to spatially varying fields and full Coulomb collisions, while retaining phase-mixing operators that couple nearby Hermite and Laguerre moments and providing a close form for the projection of the Coulomb operator in velocity space. We also note that the use of shifted velocity polynomials in the Hermite-Laguerre basis, which gives rise to the fluid operator \( F_a^{pj} \), allows us to have an efficient representation of the distribution function both in the weak \( (u_{||a} \ll v_{tha}) \) and strong flow \( (u_{||a} \sim v_{tha}) \) regimes. As we will see in Section 6, the fluid operator \( F_a^{pj} \) generates the lowest order fluid equations, as it is present even if all kinetic moments \( N_a^{pj} \) (except \( N_a^{00} \)) are set to zero.

5. Poisson’s Equation

We use Poisson’s equation to evaluate the electric field appearing in the moment hierarchy equation, Eq. (4.8). In \((x, v)\) coordinates, Poisson’s equation reads

\[
F_a^{pj} = \frac{d a^{pj}}{d t} \ln \left( N_a T_{||a} T_{\perp a} B^{-j} \right) + \frac{\sqrt{2 \beta}}{v_{th||a}} \frac{d a^{pj-1}}{d t} u_{||a} + \frac{\sqrt{p(p-1)}}{2} \frac{d a^{pj-2}}{d t} \ln T_{||a} - j \frac{d a^{pj-1}}{d t} \ln \left( \frac{T_{||a}}{B} \right),
\]
Following the same steps used to derive Eq. (3.24) from Eq. (3.20), we can write Poisson’s equation, Eq. (5.1), as

$$\epsilon_0 \nabla \cdot E = \sum_a q_a \delta a = \sum_a q_a \int f_a d^3 v. \quad (5.1)$$

Equation (5.2) shows that all particles that have a Larmor orbit crossing a given point \(x\), give a contribution to the charge density at this location.

Performing the integral over \(R\) and introducing the Fourier transform \(F_a(x - \rho_a a, v\|, \mu, \theta) = \int d^3 k F_a(k, v\|, \mu, \theta)e^{-ik\cdot x}e^{i\rho_a k\cdot a}\), Eq. (5.2) can be rewritten as

$$\epsilon_0 \nabla \cdot E = \sum_a q_a \int dv\|d\mu d\theta \frac{B^*_a}{m_a} F_a(k, v\|, \mu, \theta)e^{-ik\cdot x}e^{i\rho_a k\cdot a}. \quad (5.3)$$

To perform the \(k\) integration, we use the cylindrical coordinate system \((k, \alpha, k\|, \perp)\), expressing \(k = k\| (\cos \alpha e_1 + \sin \alpha e_2) + k\|b\), such that \(k\cdot a = k\| \cos(\theta - \alpha)\). This coordinate system allows us to express \(e^{i\rho_a k\cdot a}\) in Eq. (5.3) in terms of Bessel functions. Indeed, \(e^{i k\| \rho_a \cos(\theta - \alpha)} = J_0(k\| \rho_a) + 2 \sum_{l=1}^{\infty} I_l(k\| \rho_a) l^l \cos[l(\theta - \alpha)]\) (Abramowitz et al. 1965), where \(I_l(k\| \rho_a)\) is the Bessel function of the first kind of order \(l\). We can then write

$$\epsilon_0 \nabla \cdot E = \sum_a q_a \int dv\|d\mu d\theta \frac{B^*_a}{m_a} \left( I_0[F_a] + 2 \sum_{l=1}^{\infty} l^l I_l[F_a \cos[l(\theta - \alpha)]\right). \quad (5.4)$$

where the Fourier-Bessel operator \(I_l[f]\) is defined as

$$I_l[F_a(k, v\|, \mu, \theta)] \equiv \int d^3 k J_l(k\| \rho_a) F_a(k, v\|, \mu, \theta)e^{-ik\cdot x}. \quad (5.5)$$

Introducing the Fourier decomposition of \(\tilde{F}_a\), Eq. (2.39), in Eq. (5.4), we obtain

$$\epsilon_0 \nabla \cdot E = \sum_a q_a \int dv\|d\mu \frac{B^*_a}{m_a} \left[ I_0[\langle F_a\rangle] + 2 \pi \sum_{l=1}^{\infty} \frac{l^{l-1}}{l!} I_l[C_{la} e^{il\alpha} + C_{-la} e^{-il\alpha}]\right], \quad (5.6)$$

where the \(\theta\) integration was performed by using the identity \(\int_0^{2\pi} e^{i\theta(l-m)}d\theta = 2\pi \delta(l-m)\). Notice that \(\int_0^{2\pi} I_0[\langle F_a\rangle]d\theta/2\pi = I_0(\langle F_a\rangle)\), and corresponds to the \(J_0(k\| \rho_a)\) operator used in most gyrofluid closures (Hammett et al. 1992; Snyder & Hammett 2001; Madsen 2013), and in the gyrokinetic Poisson equation (Lee 1983; Dubin et al. 1983).

We now order the terms appearing in Eq. (5.6). Using the Taylor series expansion of a Bessel function \(J_l(x)\) of order \(l\) (Abramowitz et al. 1965), we find

$$I_0[\langle F_a\rangle] \sim \left[ 1 - \frac{(k\| \rho_a)^2}{4} + O(e^4)\right]\langle F_a\rangle, \quad (5.7)$$

while using the orderings of \(\nu_e\) and \(\nu_i\) in Eqs. (1.2) and (2.28)

$$\frac{I_l[C_{la}]}{\Omega_a} \lesssim \epsilon_o e^{l+1} \langle F_a\rangle. \quad (5.8)$$
for \( l \geq 1 \).

Consistently with Section 3.3, we neglect the \( l \geq 1 \) collisional terms, therefore representing Poisson’s equation up to \( O(\epsilon_B) \). For the derivation of an higher-order Poisson equation, the treatment of finite \( l \geq 1 \) collisional effects are presented in Appendix C. Taylor expanding \( J_0(x) \simeq 1 - x^2/4 \), Poisson’s equation reads

\[
\epsilon_0 \nabla \cdot E = \sum_a q_a \left[ N_a \left( 1 + \frac{b \cdot \nabla \times b}{\Omega_a} u_{||a} + \frac{b \cdot \nabla \times v_E}{\Omega_a} \right) + \frac{1}{2m_a} \nabla^2 \left( \frac{P_{\perp a}}{\Omega_a^2} \right) \right].
\]

(5.9)

6. Collisional Drift-Reduced Fluid Model

The infinite set of equations that describe the evolution of the moments of the distribution function, Eq. (4.8), and Poisson’s equation, Eq. (5.9), constitute the drift-reduced model, which is valid for distribution functions arbitrarily far from equilibrium. For practical purposes, a closure scheme must be provided in order to reduce the model to a finite number of equations. In this section, we derive a closure in the high-collisionality regime. For this purpose, we first state in Section 6.1 the evolution equations for the fluid moments (i.e. \( n_a, u_{||a}, T_{||a}, T_{\perp a}, Q_{||a} \) and \( Q_{\perp a} \)), that correspond to the lowest-order indices of the moment hierarchy equation. Then, in Section 6.2, we apply a prescription for the higher-order parallel and perpendicular moment equations that allows a collisional closure for \( Q_{||a} \) and \( Q_{\perp a} \) in terms of \( n_a, u_{||a}, T_{||a} \) and \( T_{\perp a} \). The nonlinear closure prescription used here, sometimes called semi-collisional closure (Zocco & Schekochihin 2011), can be employed at arbitrary collisionalities by including a sufficiently high number of moments (indeed, it was used in Loureiro et al. (2016) to consider low-collisionality regimes). It also allows us to retain the non-linear collision contributions inherent to a full-F description that may have the same size as its linear contributions, as pointed out in Catto & Simakov (2004).

6.1. Fluid Equations

We first look at the \((p,j) = (0,0)\) case of Eq. (4.8). Noting that \( \mathbf{N}_{a}^{00} = 0 \) and \( C_{ab}^{00} = 0 \), we obtain

\[
\nabla \cdot \left( \left| \mathbf{R}_a \right|^{s00}_a + \mathcal{F}_a^{00} \right) = 0.
\]

(6.1)

Evaluating \( \left| \mathbf{R}_a \right|^{s0j}_a \) in Eq. (4.10) and \( \mathcal{F}_a^{0j} \) in Eq. (4.9), for \((p,j) = (0,0)\), Eq. (6.1) yields the continuity equation

\[
\frac{d}{dt} N_a + \frac{d}{dt} \left( \frac{N_a \nabla^2 \phi}{\Omega_a B} \right) = -N_a \nabla \cdot u_{0a} - \frac{N_a \nabla^2 \phi}{\Omega_a B} \nabla \cdot U_{0a}.
\]

(6.2)

The upper convective derivative \( d^0_a/dt \), defined by

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + u_{0a} \cdot \nabla,
\]

(6.3)

is related to the guiding-center fluid velocity \( u_{0a} \)

\[
u_{0a} = U_{0a} + \frac{T_{||a} + T_{\perp a}}{m_a} \frac{b \times \nabla B}{\Omega_a B} + \frac{b}{\Omega_a} \times \frac{d_{0a} U_{0a}}{dt},
\]

(6.4)
and it differs from the lower-convective derivative $d_{0a}/dt$ in Eq. (4.4) by the addition of the last two terms in Eq. (6.4). The vorticity $\nabla_\perp^2 \phi$ is related to the $E \times B$ drift by

$$\frac{b \cdot \nabla \times \mathbf{v}_E}{\Omega_a} = \frac{\nabla_\perp^2 \phi}{B \Omega_a} + O(\epsilon^3),$$

(6.5)

and it appears in Eq. (6.2) due to the difference between $N_{a0}$ and $N_{a0}^0$ [see Eq. (4.6)]. To derive Eq. (6.2), we use the low-$\beta$ limit expression for $b \times k \simeq (b \times \nabla B)/B$ and neglect $u_{\parallel a} \cdot \nabla \times b/\Omega_a$ as

$$u_{\parallel a} \cdot \nabla \times b = \frac{T_e}{T_i} \beta \sim \epsilon^3,$$

(6.6)

therefore keeping up to $O(\epsilon^2)$ terms [namely the $\nabla_\perp^2 \phi$ term in Eq. (6.5)].

The parallel momentum equation is obtained by setting $(p, j) = (1, 0)$ in Eq. (4.8), yielding

$$m_a \frac{d_{\parallel a}^0 u_{\parallel a}}{dt} = \frac{m_a v_{th} u_{\parallel a}}{\sqrt{2}} \sum_b C_{ab}^{10} - \frac{m_a \nabla_\perp^2 \phi d_{0a} u_{\parallel a}}{\Omega_a B} \frac{dt}{dt} - \frac{m_a}{\sqrt{2} N_a} \nabla \cdot (u_{\parallel a}^1 N_a v_{th} u_{\parallel a})$$

$$+ m_a ||A||_{a0}^0 + \left(1 + \frac{\nabla_\perp^2 \phi}{\Omega_a B}\right) \left(q_a E_{\parallel} - T_{\perp a} \frac{\nabla B}{B} + m_a v_{E} \cdot \frac{d_{0a} b}{dt}\right),$$

(6.7)

with

$$u_{\parallel a}^1 = \frac{U_{pa}^{th}}{\sqrt{2}} + \sqrt{2} \left(\frac{b \times \nabla B}{\Omega_a B} Q_{\parallel a} + Q_{\perp a}\right) + v_{th} u_{\parallel a} - \frac{b}{2} \left(1 + \frac{\nabla_\perp^2 \phi}{\Omega_a B}\right).$$

(6.8)

The expression for $C_{ab}^{10}$ is given in Appendix D, as well as all the $C_{ab}^{pj}$ coefficients relevant for the present fluid model. The left-hand side of Eq. (6.7) describes the convection of $u_{\parallel a}$, while the first term in the right-hand side is related to pressure and heat flux gradients, the second term to resistivity (collisional effects), the third term consists of high-order terms kept to ensure phase-space conservation properties, and the last term is the parallel fluid acceleration, namely due to parallel electric fields, mirror force, and inertia.

The parallel and perpendicular temperature equations are obtained by setting $(p, j) = (2, 0)$ and $(0, 1)$ respectively in Eq. (4.8). This yields for the parallel temperature

$$\frac{N_a d_{\parallel a} T_{\parallel a}^0}{\sqrt{2}} = \frac{\nabla B}{B} - \frac{N_a \nabla_\perp^2 \phi d_{0a} T_{\parallel a}^0}{\sqrt{2} \Omega_a B} \frac{dt}{dt} - \frac{2 N_a T_{\parallel a}^0}{v_{th} ||A||_{a}^{10}} u_{\parallel a}^1 \cdot \nabla u_{\parallel a}$$

$$- \nabla \cdot (N_a T_{\parallel a}^0 u_{\parallel a}^2 ||A||_{a}^{10} + N_a T_{\parallel a} E_{\parallel} \cdot b \times \nabla B B) \left(1 + \frac{\nabla_\perp^2 \phi}{\Omega_a B}\right)$$

$$+ \sum_b C_{ab}^{20} N T_{\parallel a} \frac{2 N_a T_{\parallel a} ||A||_{a}^{10}}{v_{th} ||A||_{a}^{10}},$$

(6.9)

where

$$u_{\parallel a}^2 = \frac{Q_{\parallel a}}{2 N_a T_{\parallel a}} \frac{U_{pa}^{th}}{v_{th} ||A||_{a}^{10}} + \sqrt{2} \frac{T_{\parallel a} b \times \nabla B}{m_a \Omega_a B} + \frac{b}{2} \frac{Q_{\parallel a}}{2 N_a T_{\parallel a}} \left(1 + \frac{\nabla_\perp^2 \phi}{\Omega_a B}\right),$$

(6.10)

and for the perpendicular temperature
\[ N_a \frac{d \rho_a}{dt} \left( \frac{T_{\perp a}}{B} \right) + \frac{N_a \nabla^2 \phi}{\Omega_a B} \frac{d \rho_{0a}}{dt} \left( \frac{T_{\perp a}}{B} \right) = \nabla \cdot \left( \frac{N_a T_{\perp a}}{B} \frac{u_{a}^{2 \perp}}{B} \right) - \frac{N_a T_{\perp a}}{B} \sum_b C_{ab}^{01}, \]  

with

\[ u_{a}^{2 \perp} = - \frac{Q_{\perp a}}{N_a T_{\perp a}} \frac{U_{\perp a}^{th}}{v_{th} a} - \frac{T_{\perp a} b \times \nabla B}{m_a \Omega_a B}. \]  

The equations for the evolution of the parallel \( Q_{\parallel a} \) and perpendicular \( Q_{\perp a} \) heat fluxes are obtained by setting \((p, j) = (3, 0)\) and \((1, 1)\) respectively in Eq. (4.8), yielding

\[ \frac{d Q_{\parallel a}}{dt} = - \frac{d \rho_{0a}}{dt} \left( \frac{Q_{\parallel a}}{\Omega_a B} \right) \nabla^2 \phi + N_a T_{\parallel a} \sqrt{3} v_{th} |a| \sum_b C_{ab}^{30} \]

\[ - Q_{\parallel a} \nabla \cdot u_0 - \frac{Q_{\parallel a}}{\Omega_a B} \nabla \cdot U_{0a} - 3 \nabla \cdot (u_{ka} Q_{\parallel a}) \]

\[ - \frac{3}{\sqrt{2}} \left( 1 + \frac{\nabla^2 \phi}{\Omega_a B} \right) \frac{E \cdot b \times \nabla B}{B^2} Q_{\parallel a} + 3 \sqrt{2} N_a T_{\parallel a} |A|^2_{\parallel a} \]

\[ - 3 \sqrt{2} N_a T_{\parallel a} u_{a}^{2 \parallel} \cdot \nabla u_{\parallel a} - 3 \sqrt{2} N_a v_{th} a u_{a}^{\perp} \cdot \nabla T_{\parallel a}, \]

and

\[ \frac{d Q_{\perp a}}{dt} = - \frac{d \rho_{0a}}{dt} \left( \frac{Q_{\perp a}}{\Omega_a B} \right) \nabla^2 \phi + N_a v_{th} |a| \left( \frac{u_{a}^{\parallel} \cdot \nabla}{\sqrt{2}} \right) \frac{T_{\perp a}}{B} \]

\[ + \frac{N_a T_{\perp a}}{B} \left( u_{a}^{2 \perp} \cdot \nabla \right) u_{\parallel a} - \left( \frac{Q_{\perp a}}{B} \right) \left( \nabla \cdot u_0^0 + \frac{\nabla^2 \phi}{\Omega B} \nabla \cdot U_{0a} \right) \]

\[ - (U_{ka} + 2 U_{\parallel B}) \cdot \nabla \left( \frac{Q_{\perp a}}{B} \right) - \frac{\sum_b C_{ab}^{11} v_{th} |a| N_a T_{\perp a}}{B} \]

\[ + \left( \frac{N_a T_{\perp a} \nabla B}{m_a B^2} + \frac{Q_{\perp a}}{B} \frac{E \cdot b \times \nabla B}{B^2} \right) \left( 1 + \frac{\nabla^2 \phi}{\Omega_a B} \right). \]

In Eqs. (6.13) and (6.14) we neglected the higher-order moments with respect to \( N_{30} \) and \( N_{11} \), an approximation that we will scrutinize in the next section. Equations (6.2)-(6.14) constitute a closed set of six coupled non-linear partial differential equations for both the hydrodynamical variables \( n_a, u_{\parallel a}, T_{\parallel a}, T_{\perp a} \), and the kinetic variables \( Q_{\parallel a} \) and \( Q_{\perp a} \).

With respect to previous delta-F (Dorland & Hammert 1993; Brizard 1992) and full-F gyrofluid models (Madsen 2013), our fluid model, Eqs. (6.2-6.14), while neglecting \( k_{\perp r} \) effects, includes the velocity contributions from the \( B_{\|}^* \) denominator in the equations of motion (2.15) and (2.16) and includes the effects of full Coulomb collisions up to order \( \epsilon_c \). Also, due to the choice of basis functions with shifted velocity arguments \( H_p(s_{\| a}) \) instead of \( H_p(v_{\| a}/v_{th a}) \), we obtain a set of equations that can efficiently describe both weak flow \( (u_{\parallel a} \ll v_{th a}) \) and strong flow \( (u_{\parallel a} \sim v_{th a}) \) regimes.

### 6.2. High Collisionality Regime

We now consider the high-collisionality regime, where the characteristic fluctuation frequency of the hydrodynamical variables \( \omega \)
analogous expressions are obtained for the ion species. according to eq. (6.18), all moments where the mean free path is much smaller than the collision frequency is much smaller than the collision frequency, that is

$$\delta_a \sim \frac{\omega}{\nu_a} \sim \frac{\lambda_{mfp}}{L_{||a}} \ll 1,$$

where the mean free path \(\lambda_{mfp}\) in eq. (6.16) is defined as

$$\lambda_{mfp} = \frac{v_{tha}}{\nu_{aa}}.$$

Equation (6.16) describes the so-called linear transport regime (Balescu 1988). In this case, the distribution function can be expanded around a Maxwell-Boltzmann equilibrium, according to the Chapman-Enskog asymptotic closure scheme (Chapman 1962) and, to first order in \(\delta_a\), we have

$$\langle F_a \rangle \simeq F_{Ma} \left[1 + \delta_a f_{1a}(v||, \mu, R, t)\right].$$

According to eq. (6.18), all moments \(N_a^q\) in the Hermite-Laguerre expansion eq. (3.1) with \((p, j) \neq (0, 0)\) are order \(\delta_a\). Since \(Q_{||a}\) and \(Q_{\perp a}\) are determined at first order in \(\delta_a\) only by the moments \((p, j) = (0, 0), (3, 0), (1, 1),\) the truncation of sec. (6.1), i.e., neglecting \((p, j) \neq (0, 0), (3, 0), (1, 1)\) is justified. for a more detailed discussion on this topic see Balescu (1988). Moreover, in the linear regime, a relationship between the hydrodynamical and kinetic variables can be obtained along the lines of the semi-collisional closure. This allows us to express \(Q_{||a}\) and \(Q_{\perp a}\) as a function of \(N_a, u_{||a}, T_{||a}\) and \(T_{\perp a}\), therefore reducing the number of equations. We now derive this functional relationship.

We consider eqs. (6.13)-(6.14) in the linear regime, and neglect the polarization terms that are proportional to \(\nabla^2 \phi / (\Omega_a B)\). This yields \(\sqrt{3/2} \sum_b C_{ab}^{30} / v_{th||a} \simeq R_{||a}\) and \(\sum_b C_{ab}^{11} / (\sqrt{2} v_{th||a}) \simeq R_{\perp a}\), with \(R_{||a}\) and \(R_{\perp a}\) given by

$$R_{||a} = \frac{\nabla || T_{||a}}{T_{||a}} + u_{||a} = \frac{b \times \nabla B}{\Omega_a B} \cdot \left(\frac{\nabla u_{||a}}{u_{||a}} + \frac{\nabla T_{||a}}{T_{||a}}\right),$$

$$R_{\perp a} = \frac{T_{\perp a} \nabla \perp B}{B} - \frac{1}{2 \sqrt{2}} \nabla || T_{\perp a} - u_{||a} = \frac{b \times \nabla B}{\Omega_a B} \cdot \left(\frac{T_{\perp a} \nabla u_{||a}}{T_{||a} u_{||a}} + \nabla \ln \frac{T_{\perp a}}{T_{||a}}\right),$$

since \(d_{a} / dt \sim d_{0a} / dt \sim \omega\) and \((d^0 Q_{||||a} / dt) / Q_{||||a} \sim \delta_a^2 \nu_a\). We compute the guiding-center moments of the collision operator \(C_{ab}^{30}\) and \(C_{ab}^{11}\) by truncating the series for the like-species collision operator in eq. (3.40) at \((l, k, n, q) = (2, 1, 2, 1)\). The resulting \(C_{ab}^{pq}\) coefficients are presented in appendix D.

With the expression of \(C_{ab}^{30}\) and \(C_{ab}^{11}\), we can solve for \(Q_{||a}\) and \(Q_{\perp a}\). In the regime \((T_{||a} - T_{\perp a}) / T_a \sim \delta_a\), at lowest order, we obtain for the electron species

$$\frac{Q_{||e}}{N_e T_e v_{the}} = -0.362 \left(\frac{u_{||e} - u_i}{v_{the}}\right) - 10.6 \lambda_{mfp} \frac{\nabla || T_e}{T_e},$$

and

$$\frac{Q_{\perp e}}{N_e T_e v_{the}} = -0.119 \left(\frac{u_{||e} - u_i}{v_{the}}\right) - 3.02 \lambda_{mfp} \frac{\nabla || T_e}{T_e},$$

analogous expressions are obtained for the ion species.
Equations (6.2), (6.7), (6.9), and (6.11), with $Q_{∥a}$ and $Q_{⊥a}$ given by Eqs. (6.21) and (6.22) are valid in the high-collisionality regime, and can be compared with the drift-reduced Braginskii equations in Zeiler et al. (1997). We first rewrite the continuity equation, Eq. (6.2), in the form

$$\frac{∂N_e}{∂t} + \nabla \cdot \left[ N_e \left( v_E + u_{∥e} b + \frac{T_{∥e} + T_{⊥e}}{m_e} \frac{b \times \nabla B}{\Omega_e B} \right) \right] = 0,$$

(6.23)

where we expand the convective derivative $d^0 a/dt$ using Eq. (6.3) and Eq. (6.4), and neglect polarization terms proportional to the electron mass $m_e$. By noting that the diamagnetic drift $v_{de}$ can be written as

$$v_{de} = \frac{1}{e N_e} \nabla \times \frac{p_e b}{B} - 2 \frac{T_e}{m_e} \frac{b \times \nabla B}{\Omega_e B},$$

(6.24)

and by considering the isotropic regime $T_{∥e} \sim T_{⊥e} \sim T_e$, we obtain

$$\frac{∂N_e}{∂t} + \nabla \cdot \left[ N_e (v_E + u_{∥e} b + v_{de}) \right] = 0,$$

(6.25)

which corresponds to the continuity equation in the drift-reduced Braginskii model in Zeiler et al. (1997). In that model, the polarization equation is obtained by subtracting both electron and ion continuity equations, using Poisson’s equation $n_e \simeq n_i$ with $n_e$ and $n_i$ the electron and ion particle densities respectively, and neglecting the electron to ion mass ratio. Applying the same procedure to the present fluid model, we obtain

$$0 = \nabla \cdot \left( \frac{\nabla^2 \phi N_i u_{||i} b}{\Omega_i B} \right) - \nabla \cdot \left[ \frac{v_E}{2m_i} \nabla^2 \left( \frac{N_i T_{||i}}{\Omega_i^2} \right) \right] - \frac{1}{2m_i} \frac{∂}{∂t} \nabla^2 \left( \frac{N_i T_{||i}}{\Omega_i^2} \right)$$

$$+ \nabla \cdot \left( \frac{N_i}{\Omega_i^2} b \times \frac{d_0 i U_{0i}}{dt} \right) + \nabla \cdot \left[ b \left( N_i u_{||i} - N_e u_{||e} \right) \right]$$

$$+ \nabla \cdot \left( N_i T_{||i} + N_e T_{||e} + N_i T_{⊥i} + N_e T_{⊥e} \right) \frac{b \times \nabla B}{e B^2}.$$ 

(6.26)

In Eq. (6.26), the first three terms, which are not present in the drift-reduced Braginskii model, correspond to the difference between ion guiding-center density $N_i$ and particle density $n_i$, proportional to both $\nabla^2 \phi$ and $\nabla^2 P_i$. The parallel momentum and temperature equations, Eq. (6.7) and Eq. (6.9), with respect to (Zeiler et al. 1997), contain the higher-order term $A$ that ensures phase-space conservation, mirror force terms proportional to $(\nabla || B)/B$, and polarization terms proportional to $\nabla^2 \phi/(\Omega_a B)$ due to the difference between guiding-center and particle fluid quantities. This set of fluid equations constitute an improvement over the drift-reduced Braginskii model. With respect to the original Braginskii equations (Braginskii 1965), they include the non-linear terms that arise when retaining full Coulomb collisions, and the effect of ion-electron collisions.

### 7. Conclusion

In the present work, a full-F drift-kinetic model is developed, suitable to describe the plasma dynamics in the SOL region of tokamak devices at arbitrary collisionality. Taking advantage of the separation between the turbulent and gyromotion scales, a gyroaveraged Lagrangian and its corresponding equations of motion are obtained. This is the starting point to deduce a drift-kinetic Boltzmann equation with full Coulomb collisions for the gyroaveraged distribution function.
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The gyroaveraged distribution function is then expanded into an Hermite-Laguerre basis, and the coefficients of the expansion are related to the lowest-order gyrofluid moments. The fluid moment expansion of the Coulomb operator described in Ji & Held (2009) is reviewed, and its respective particle moments are written in terms of coefficients of the Hermite-Laguerre expansion, relating both expansions. This allows us to express analytically the moments of the collision operator in terms of guiding-center moments. A set of equations, indeed a moment hierarchy that describes the evolution of the guiding-center moments is derived, together with a Poisson’s equation accurate up to $\epsilon^2$. These are then used to derive a fluid model in the high-collisionality limit.

The drift-kinetic model derived herein can be considered a starting point for the development of a gyrokinetic Boltzmann equation suitable for the SOL region (e.g. Qin et al. (2007); Hahm et al. (2009)). Indeed, using a similar approach, a gyrokinetic moment hierarchy may be derived, allowing for the use of perpendicular wave numbers satisfying $k\perp \rho_s \sim 1$. For a recent Hermite-Laguerre formulation of the non-linear delta-F gyrokinetic equation see Mandell et al. (2017).

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Appendix A. Basis Transformation

In the present Appendix, we derive the expressions for the coefficients $T_{aik}^{pj}$ appearing in Eq. (3.30). These coefficients allows us to express up to order $\epsilon c_\rho$ the relation between fluid $M_a^{ik}$ and guiding-center $N_a^{ik}$ moments via Eq. (3.32). As a first step, we define a transformation similar to Eq. (3.30) but with isotropic temperatures between both bases

$$c_a^l P_l (\xi_a) L_k^{l+1/2} (c_a^2) = \sum_{p=0}^{l+2k} \sum_{j=0}^{l+[l/2]} \bar{T}_{lka}^{pj} H_p \left( \frac{v_{\parallel} - u_{\parallel} a}{v_{tha}} \right) L_j \left( \frac{v_{\perp}^2}{v_{tha}^2} \right), \quad (A1)$$

with the inverse transformation

$$H_p \left( \frac{v_{\parallel} - u_{\parallel} a}{v_{tha}} \right) L_j \left( \frac{v_{\perp}^2}{v_{tha}^2} \right) = \sum_{l=0}^{p+2j} \sum_{k=0}^{j+[p/2]} \left( T^{-1} \right)_{pja}^{lk}$$

$$\times c_a^l P_l (\xi_a) L_k^{l+1/2} (c_a^2), \quad (A2)$$

The relation between the coefficients $\left( T^{-1} \right)_{pja}^{lk}$ and $\bar{T}_{lka}^{pj}$ is given by

$$\left( T^{-1} \right)_{pja}^{lk} = \frac{\sqrt{\pi} 2^p p! (l + 1/2)!}{(k + l + 1/2)!} \bar{T}_{lka}^{pj}. \quad (A3)$$

By integrating both sides of Eq. (A1) over the whole velocity space, the expression for $\bar{T}_{lka}^{pj}$ is obtained.
with the phase-mixing term

\[
\mathcal{T}_{pq}^{ij} = \sum_{q=0}^{\lfloor l/2 \rfloor} \sum_{p=0}^{\lfloor |p|/2 \rfloor} \sum_{r=0}^{k} \sum_{s=0}^{q} \sum_{m=0}^{\min(j,i)} \sum_{v=0}^{\lfloor k-r \rfloor} \sum_{w=0}^{\lfloor k-m \rfloor} \frac{(-1)^{q+i+j+v+m}}{2^{\lfloor \frac{k-r}{2} \rfloor + \lfloor \frac{k-m}{2} \rfloor} + v} \\
\times \left( \begin{array}{c} \frac{1}{l} \\
\frac{1}{l} \\
\frac{1}{l} \\
\frac{1}{l} \\
\frac{1}{l} \\
\end{array} \right) \left( \begin{array}{c} \frac{1}{r} \\
\frac{1}{r} \\
\frac{1}{r} \\
\frac{1}{r} \\
\frac{1}{r} \\
\end{array} \right) \left( \begin{array}{c} \frac{1}{s} \\
\frac{1}{s} \\
\frac{1}{s} \\
\frac{1}{s} \\
\frac{1}{s} \\
\end{array} \right) r! \\
\times \frac{(k-i+l-1/2)!(l+p+2(m-r-v)-1)!}{(p-2v)!(k-i-m)!(l+m-1/2)! v! m!}.
\] (A4)

We then integrate both sides of Eq. (3.30) with weights \( H_{l}(s_{\parallel a}) \) \( L_{j}(s_{\perp a}^{2}) \), with the argument transformation

\[
H_{p}(s_{\parallel a}) = \left( \frac{T_{a}}{T_{\parallel a}} \right)^{p/2} \sum_{k=0}^{\lfloor p/2 \rfloor} \frac{p!}{k!(p-2k)!} \left( 1 - \frac{T_{\parallel a}}{T_{a}} \right)^{k} H_{p-2k} \left( \frac{v_{\parallel} - u_{\parallel a}}{v_{\parallel a}} \right),
\] (A5)

and

\[
L_{j}(s_{\perp a}^{2}) = \sum_{k=0}^{j} \left( \begin{array}{c} j \\
-j \\
\end{array} \right) \left( \frac{T_{a}}{T_{\perp a}} \right)^{k} \left( 1 - \frac{T_{\parallel a}}{T_{a}} \right)^{j-k} L_{k} \left( \frac{v_{\perp}^{2}}{v_{\perp a}} \right),
\] (A6)

to find the relation between the isotropic and anisotropic temperature coefficients

\[
T_{al}^{pq} = \sum_{pp=0}^{l+2k} \sum_{jj=0}^{k+\lfloor l/2 \rfloor} \sum_{z=0}^{jj} \sum_{d=0}^{\lfloor jj/2 \rfloor} \left( \begin{array}{c} jj \\
jj \\
\end{array} \right) \frac{pp!d_{r-j} \delta_{j,p-2d}}{d!(p-2d)!} \\
\times \left( \frac{T_{\parallel a}}{T_{a}} \right)^{p/2} \left( \frac{T_{\perp a}}{T_{a}} \right)^{z} \left( 1 - \frac{T_{\parallel a}}{T_{a}} \right)^{d} \left( 1 - \frac{T_{\perp a}}{T_{a}} \right)^{j-z} \mathcal{T}_{pq}^{ij},
\] (A7)

\[
\left( T_{a}^{-1} \right)^{lpq} = \sum_{z=0}^{j} \sum_{d=0}^{p-2d+2z} \sum_{ll=0}^{d-\lfloor |p|/2 \rfloor} \sum_{kk=0}^{z} \sum_{kk=0}^{l} \left( \begin{array}{c} j \\
-j \\
\end{array} \right) \frac{p!d_{r-j} \delta_{j,k,k}}{d!(p-2d)!} \\
\times \left( \frac{T_{a}}{T_{\parallel a}} \right)^{p/2} \left( \frac{T_{a}}{T_{\perp a}} \right)^{z} \left( 1 - \frac{T_{\parallel a}}{T_{a}} \right)^{d} \left( 1 - \frac{T_{\perp a}}{T_{a}} \right)^{j-z} \left( T_{a}^{-1} \right)^{lpq}.
\] (A8)

**Appendix B. Guiding-Center Moments of A**

In Eq. (2.16), the term \( A \) that ensures phase-space conservation properties for the particle equations of motion is introduced. Here, we present the analytic expressions for its guiding-center moments \( ||A||_{a}^{lpq} \) appearing in Eq. (4.11). These are given by

\[
||A||_{a}^{lpq} = \frac{1}{N_{a}/\Omega_{a}} \sum_{l,k} \left( A_{a1} \mathcal{V}_{l}^{lpq} + A_{a2} \mathcal{V}_{l}^{lpq} + A_{a3} \mathcal{V}_{l}^{lpq} \right.
\]

\[
+ A_{a4} \mathcal{V}_{l}^{lpq} \mathcal{M}_{l}^{lpq} + A_{a5} \mathcal{M}_{l}^{lpq} + A_{a6} \mathcal{P}_{l}^{lpq} \right) N_{a}^{lpq},
\] (B1)

with the phase-mixing term.
\[ V^{3pj}_{lk} = \left[ \sqrt{(p+3)(p+2)(p+1)}\delta_{p+3,l} + 3\sqrt{(p+1)^3}\delta_{p+1,l} \right] \\
+ 3\sqrt{p^3}\delta_{p-1,l} + \sqrt{p(p-1)(p-2)}\delta_{p-3,l} \right] \frac{\delta_{j,k}}{\sqrt{8}}, \]  
\hspace{5cm} (B2)

and the coefficients \( A_i \)

\[ A_{1a} = v_{th}^3 a \| k \cdot \nabla \times b, \]  
\hspace{5cm} (B3)

\[ A_{2a} = v_{th}^2 a \| \left( [k \cdot (u \| a \nabla \times b + \nabla \times v_E) + \nabla \times b \cdot A] \right), \]  
\hspace{5cm} (B4)

\[ A_{3a} = v_{th} a \| (u \| a \nabla \times b + \nabla \times v_E) \cdot A + v_{th}^2 a \| \nabla \times b \cdot C, \]  
\hspace{5cm} (B5)

\[ A_{4a} = v_{th} a \| \frac{T_i}{ma B} \nabla \perp B \cdot \nabla \times b, \]  
\hspace{5cm} (B6)

\[ A_{5a} = \frac{T_{1a} a}{ma B} \nabla \perp B \cdot (u \| a \nabla \times b + \nabla \times v_E), \]  
\hspace{5cm} (B7)

\[ A_{6a} = (v_{th} a \| a \nabla \times b + \nabla \times v_E) \cdot C \]  
\hspace{5cm} (B8)

with

\[ A = \left[ \frac{\partial b}{\partial t} + (b \cdot \nabla) v_E + (v_E \cdot \nabla) b + 2u \| a v_{th} a \| k \right] \perp, \]  
\hspace{5cm} (B9)

\[ C = \frac{1}{v_{th} a} \left[ \frac{\partial v_E}{\partial t} + (v_E \cdot \nabla) v_E + u^2 \| a k \right] \perp. \]  
\hspace{5cm} (B10)

**Appendix C. Poisson’s Equation with Collisional Effects**

To include \( \epsilon_\nu \) effects in Poisson’s equation, we retain the \( l = 1 \) Bessel term in Eq. (5.6), yielding

\[ \epsilon_0 \nabla \cdot E = \sum_a q_a \left[ N_a \left( 1 + \frac{b \cdot \nabla \times b}{\Omega} u \| a + \frac{b \cdot \nabla \times v_E}{\Omega} \right) \right] \\
+ \frac{1}{2m_a} \nabla^2 \left( \frac{P_{1a}}{\Omega a^2} \right) + 2\pi \int \Gamma_1 \left[ C_{1a} e^{i\alpha} + C_{-1a} e^{-i\alpha} \right] \frac{B_m^\ast}{m_a} dv \| d\mu. \]  
\hspace{5cm} (C1)

The collisional terms \( C_{\pm 1a} = \sum_b C_{\pm 1ab} \) (for collisions between species \( a \) and \( b \)) can be cast in terms of gyrofluid moments \( N_{lk}^a \). For like-species collisions, we use Eq. (3.22) to express the collision operator \( C_{aa0} \) in Eq. (3.21) in terms of fluid moments, together with the property (Ji & Held 2006)

\[ P^i(v) \cdot T^{lk} = v^l \cdot T^{lk}, \]  
\hspace{5cm} (C2)

which holds for any totally symmetric and traceless tensor \( T^{lk} \). This yields the following form for the lowest-order collision operator Eq. (3.22)
$$c_0 (f_{a,km}, f_{a,nqr}) = f_{a,M} N_{a}^{kl} N_{a}^{nr} \sum_{u=0}^{\min(2,l,n)} \nu_{saaau}^{l,m,nr} (c^2)$$

$$\times \sum_{i=0}^{\min(l,n)-u} d_{i}^{l-u,n-u} c_{a}^{l+n-2(i+u)} \frac{\sigma_{i}^{l+n-2(i+u)}}{c_{a}^{l+n-2(i+u)}} \cdot \mathcal{P}^{l}(b) \cdot i^{l+u} \mathcal{P}^{n}(b),$$

(C3)

as $\mathcal{T}$ means the traceless symmetrization of $T$. The shifted velocity vector $c_{a} = (v - u_{a}) / v_{tha}$, under the transformation of Eq. (2.3), with the lowest-order fluid velocity $u_{a} \simeq u_{\parallel a} b + v_{E}$, can be written as

$$c_{a} = c_{\parallel a} b + \frac{c_{\perp a}}{2} (e^{i\theta} E_1 + e^{-i\theta} E_2),$$

(C4)

where $E_{1,2} = e_{2} \pm i e_{1}$. Using the multinomial theorem

$$\left( \sum_{i=1}^{m} x_{i} \right)^{k} = \sum_{a_{i} \geq 0} \frac{k!}{\prod_{i=1}^{m} a_{i}!} x_{i}^{a_{i}},$$

subject to the constraint $\sum_{i=1}^{m} a_{i} = k$, we obtain

$$c_{a}^{k} = \sum_{a_{1}+a_{2}+a_{3} = k} \frac{k!(c_{\parallel a})^{a_{1}}}{a_{1}! a_{2}! a_{3}!} \left( \frac{c_{\perp a}}{2} \right)^{a_{2}+a_{3}} e^{i\theta(a_{2}-a_{3})} b_{a_{1}} e_{a_{2}E_{1}} e_{a_{3}E_{2}}.$$  

(C5)

We use Eq. (C6) to explicit the dependence of $c_{a}$ on $\theta$ in Eq. (C3). The Fourier components $C_{1a}$ and $C_{-1a}$ correspond to the case $a_{2} = a_{3} = 1$ in the sum in Eq. (C6) above, yielding

$$C_{\pm1a} = \sum_{l,k,m,n,q,r} \frac{I_{l}^{l_{km}} L_{q}^{q_{nm}}}{\sqrt{\sigma_{k}^{l} \sigma_{q}^{q_{nm}}}} f_{a,M} N_{a}^{kl} N_{a}^{nr} \sum_{u=0}^{\min(2,l,n)} \nu_{saaau}^{l,m,nr} (c_{a}^{2}) \sum_{i=0}^{\min(l,n)-u} d_{i}^{l-u,n-u}$$

$$\times \sum_{a_{1}+a_{2}+a_{3} = k+2(l+n)+i+u} \frac{(l+n-2(i+u))!}{a_{1}! a_{2}! a_{3}!} \left( \frac{c_{\parallel a}}{2} \right)^{a_{2}+a_{3}} \delta_{a_{2},a_{3} \pm 1} \mathcal{P}^{l}(b) \cdot i^{l+u} \mathcal{P}^{n}(b).$$

(C6)

Assembling the velocity dependent terms of Eq. (C7), together with $J_{1}(k_{\perp} \rho_{a}) \simeq k_{\perp} v_{tha} c_{\perp a} / (2 \Omega_{a})$, the velocity integration of the like-species operator in Poisson’s Eq. (C1) is then

$$I_{a}^{\pm} = \int c_{\parallel a} C_{\pm1a} \frac{B_{\parallel}}{m_{a}} dv_{\parallel} d\mu = \sum_{l,k,m,n,q,r} \sum_{u=0}^{\min(2,l,n)} \sum_{i=0}^{\min(l,n)-u} \sum_{a_{1}+a_{2}+a_{3} = l+n-2(i+u)}$$

$$\times \frac{I_{l}^{l_{km}} L_{q}^{q_{nm}}}{\sqrt{\sigma_{k}^{l} \sigma_{q}^{q_{nm}}}} f_{a,M} N_{a}^{kl} N_{a}^{nr} d_{i}^{l-u,n-u} \frac{(l+n-2(i+u))!}{a_{1}! a_{2}! a_{3}! 2^{a_{2}+a_{3}}} \mathcal{P}^{l}(b) \cdot i^{l+u} \mathcal{P}^{n}(b) v_{tha}^{3} I_{a}^{\pm},$$

(C7)

where
\[ I_{a \pm} = \int f_{Ma} c_{\perp a} (c_{\| a})^{a_1} (c_{\perp a})^{a_2 \pm 1} \frac{\nu_{lm,nr}^{\perp a_{aa}} (\xi_a)}{c_{l+n-2(i+u)}} \frac{B_\|}{m_a} d\nu d\mu. \]  

(C9)

Converting to pitch angle coordinates \( v_\| = v_{tha} c_{\| a} = v_{tha} c_{\perp a} \), \( v_\perp = v_{tha} c_{\perp a} = c_\perp^2 (1 - \xi_a^2) \), with the volume element \( (B_\|/m_a) d\mu d\nu = v_{tha}^3 c_{\perp a}^2 dv \xi_a \), the integral of Eq. (C9) can be performed analytically, yielding

\[ I_{a \pm} = \int \xi_a^{a_1} (1 - \xi_a^2)^{a_3 + \sigma \pm 1} \xi_a \int f_{Ma}^{\nu_{lm,nr}^{\perp a_{aa}}} c_{\perp a}^{2(a_3 + \sigma_\pm + i + u) + a_1 - l - n} \frac{d^3 c}{4\pi} \]

\[ = \frac{(-1)^{a_1 + 1} \Gamma \left( \frac{a_1 + 1}{2} \right) \Gamma (a_3 + \sigma_\pm + 1)}{8\pi \Gamma \left( \frac{a_1}{2} + a_3 + \sigma_\pm + \frac{3}{2} \right)} C_{a_{aa}}^{a_1 - l - n} a_3 + \sigma_\pm + i + u, lm, nr, \]

where \( \sigma_\pm = (1 \pm 1)/2 \) for \( C_{\pm1} \) respectively.

For electron-ion collisions, we take the expression for \( C_{ei} = C_{ei}^0 + C_{ei}^1 \) given by Eqs. (2.25) and (2.26) respectively. By using the shifted velocity vector \( c_a \) of Eq. (C4), we can proceed as for the like-species operator, yielding the Fourier components \( C_{\pm1} = C_{\pm1}^0 + C_{\pm1}^1 \), with

\[ C_{\pm1}^0 = - \sum_{l,k} \frac{n_1 L_{ei}}{8\pi c_e^3} \frac{l(l + 1) f_{eM} N_{e}^{\nu_{l}^{lk}}}{\sqrt{\sigma_k^l}} \]

\[ \times \sum_{a_1 + a_2 + a_3 = l} \frac{l! (c_{\| e})^{a_1}}{a_1! a_2! a_3!} \left( c_{\perp e} \right)^{a_2 + a_3} b \nu_{ei} N_{e}^{\nu_{l}^{lk}} / 2 (b) \delta_{a_2, a_3 \pm 1}, \]

and

\[ C_{\pm1}^1 = f_{Me} \frac{n_1 L_{ei} m_e c_{\perp e}}{8\pi c_e^3 T_e} \frac{1}{2} u_{ei} \cdot E_{1,2}. \]  

(C12)

The velocity integration of \( C_{\pm1}^0 \) and \( C_{\pm1}^1 \) are given by

\[ I_{ei}^{0 \pm} = \int c_{\perp e} C_{\pm1}^{0 \pm} \frac{B_\|}{m_a} dv d\mu \]

\[ = - \sum_{l,k} \sum_{a_1 + a_2 + a_3 = l} \delta_{a_2, a_3 \pm 1} \nu_{ei} n_e v_{the}^2 N_{e}^{\nu_{l}^{lk}} / 16\pi^{3/2} \sqrt{\sigma_k^l} g_{a_2 + a_3} a_1! a_2! a_3! \]

\[ \times \frac{\Gamma \left( \frac{a_1 + a_2 + \sigma_\pm}{2} \right) \Gamma (a_3 + \sigma_\pm + 1)}{\Gamma \left( \frac{3}{2} + \frac{a_1}{2} + a_3 + \sigma_\pm \right)}, \]

and

\[ I_{ei}^{1 \pm} = \int c_{\perp e} C_{\pm1}^{1 \pm} \frac{B_\|}{m_a} dv d\mu = \frac{n_e} {6\sqrt{\pi}} v_{the}^2 \frac{1}{v_e} (u_{ei} \cdot E_{1,2}). \]  

(C14)

respectively. Ion-electron collisions are neglected due to the smallness of the electron to ion mass ratio. Poisson's equation including \( \epsilon^2 \) and \( \epsilon_\nu \) effects then reads
\[
\epsilon_0 \nabla \cdot E = \sum_a q_a \left[ N_a \left( 1 + \frac{b \cdot \nabla \times b}{\Omega_a} u_{|a} + \frac{b \cdot \nabla \times \mathbf{v}_E}{\Omega_a} \right) \right. \\
+ \left. \frac{1}{2m_a} \nabla_\perp^2 \left( \frac{P_{\perp a}}{\Omega_a^2} \right) + \sum_b \frac{\pi v_{tha}}{\Omega_a} \int k_\perp \left( e^{i\alpha I_{ab}^+} + e^{-i\alpha I_{ab}^-} \right) e^{i\mathbf{k} \cdot \mathbf{x}} d^3k \right],
\]

\[\text{(C15)}\]

**Appendix D. Expressions for the Moments of the Collision Operator**

In the present Appendix, we present the expressions for the guiding-center moments of the collision operator relevant for the fluid model in Section 6. The collision operator moments satisfy particle conservation

\[C_{00}^{ab} = 0,\]  \[\text{(D1)}\]

and momentum conservation

\[C_{10}^{aa} = 0,\]  \[\text{(D2)}\]

\[C_{10}^{ie} = -\frac{m_i v_{thi}}{m_e v_{thie}} C_{10}^{ie},\]  \[\text{(D3)}\]

Both the like-species and electron-ion satisfy energy conservation exactly, while the ion-electron operator satisfies Eq. (D4) at zeroth order in \(\delta_a\)

\[T_{|a} C_{20}^{ab} - \sqrt{2} T_{\perp a} C_{01}^{ab} = 0.\]  \[\text{(D4)}\]

The remaining moments \(C_{ab}^{pj}\), in the linear transport regime with \(\Delta T_{|a}/T_a = (T_{|a} - T_{\perp a})T_a \sim N^{11} \sim N^{30} \sim (u_{|e} - u_{|i})/v_{the} \sim \delta_a\), for ion-electron collisions are given by

\[C_{ie}^{10} = -\frac{m_e v_{thi}}{m_i v_{thie}} C_{ei}^{10},\]  \[\text{(D5)}\]

\[C_{ie}^{20} = \sqrt{2} \nu_{ei} \frac{m_e}{m_i} \left( \frac{T_e - T_i}{T_i} \right) - 2\sqrt{2} \nu_{ei} \frac{m_e T_e}{m_i T_i} \Delta T_i,\]  \[\text{(D6)}\]

\[C_{ie}^{01} = -2 \nu_{ei} \frac{m_e}{m_i} \left( \frac{T_e - T_i}{T_i} \right) - 2 \nu_{ei} \frac{m_e T_e}{m_i T_i} \Delta T_i,\]  \[\text{(D7)}\]

\[C_{ie}^{30} = -\nu_{ei} \sqrt{\frac{3}{2}} \frac{m_e}{m_i} \frac{Q_{|i}}{n_i T_i v_{thi}},\]  \[\text{(D8)}\]

\[C_{ie}^{11} = 3 \nu_{ei} \frac{m_e}{m_i} \frac{Q_{\perp i}}{n_i T_i v_{thi}},\]  \[\text{(D9)}\]

for electron-ion collisions.
\[ C_{ei}^{10} = \frac{\sqrt{2} \nu_{ei} u_{\parallel e} - u_{\parallel i}}{6\pi^{3/2}} v_{\text{the}} + \frac{\sqrt{2} \nu_{ei} Q_{\parallel e} + 2Q_{\perp e}}{10\pi^{3/2}} nT_e v_{\text{the}}, \]  
(D10)

\[ C_{ei}^{20} = -\frac{2\sqrt{2} \nu_{ei} \Delta T_e}{15\pi^{3/2}} T_e, \]  
(D11)

\[ C_{ei}^{30} = \frac{\sqrt{3} \nu_{ei} u_{\parallel e} - u_{\parallel i}}{10\pi^{3/2}} v_{\text{the}} - \frac{\nu_{ei}}{70\sqrt{3}\pi^{3/2}} \frac{31Q_{\parallel e} - 2Q_{\perp e}}{nT_e v_{\text{the}}}, \]  
(D12)

\[ C_{ei}^{11} = \frac{\nu_{ei}}{5\sqrt{2}\pi^{3/2}} \frac{u_{\parallel e} - u_{\parallel i}}{v_{\text{the}}} + \frac{\nu_{ei}}{150\sqrt{2}\pi^{3/2}} \frac{Q_{\parallel e} - 94Q_{\perp e}}{nT_e v_{\text{the}}}, \]  
(D13)

and for like-species collisions

\[ C_{aa}^{20} = 0, \]  
(D15)

\[ C_{aa}^{30} = -\frac{2\sqrt{2}}{125\sqrt{3}\pi^{3/2}} \frac{\nu_{aa}}{nT_a v_{\text{tha}}} (19Q_{\parallel a} - 7Q_{\perp a}), \]  
(D16)

\[ C_{aa}^{11} = -\frac{2}{375\pi^{3/2}} \frac{\nu_{aa}}{nT_a v_{\text{tha}}} (7Q_{\parallel a} - 121Q_{\perp a}). \]  
(D17)

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