One-Dimensional Full-Wave Analysis of Reflectometry Sensitivity and Correlations

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One-Dimensional Full-Wave Analysis of Reflectometry Sensitivity and Correlations

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Abstract

A general analysis is presented of the sensitivity of reflectometry to perturbations of the plasma profile, using a full-wave description in one-dimension. The square of the wavenumber is allowed to have an imaginary part, to account for absorption or divergence of the wave. Correlation reflectometry is investigated. It is found that the phase correlation is substantial, regardless of the correlation length of the fluctuations, unless either the wave attenuation is substantial or else the fluctuation correlation function is non-monotonic, corresponding to narrow-band turbulence. Curves are presented that provide fairly general information for purposes of experimental interpretation.
1. Introduction

Reflectometry has recently gained considerable attention as a diagnostic in large tokamaks (e.g. Simonet, 1985, Hubbard et al, 1987). It has both advantages and disadvantages as a measurement of the density profile, when compared to the more establish technique of interferometry (Hutchinson, 1987). However, a major impetus in the application of reflectometry is the observation that it is highly sensitive to density fluctuations (e.g. TFR Group, 1985). This fact, combined with its advantages for measurements at the plasma edge, where the fluctuations tend to be largest, has led to hopes that it may provide a powerful diagnostic of the fluctuations that are thought to be responsible for transport. These hopes have been confirmed to some extent, in a general way, by observations of large changes in the reflectometry fluctuations during changes in plasma confinement mode on tokamak DIII-D (Doyle et al, 1990). However, obtaining more specific information than simply the general level of fluctuations has proven more difficult.

Recently, experiments have been conducted forming correlations between signals from reflectometers operating at adjacent frequencies (Cripwell et al, 1989, Hanson et al, 1990). The interpretation of these experiments seems rather difficult, especially since quite surprising results have been observed. Indeed, simulation experiments and code development have been begun (Baang, et al, 1990) in an attempt to try to understand these types of experiments.

The purpose of the present work is to present in a systematic way a full-wave analysis of the sensitivity of reflectometry to changes in the density, including, therefore, fluctuations. The analysis is one-dimensional. However, some of the features of the multi-dimensional situation, notably the wave attenuation due to beam divergence, are modelled by adjustments to the one-dimensional equations.

The analysis is based on what amounts to the first Born approximation. This approach to the reflectometry problem has a long history, dating back at least to Pitteway (1959). However, it differs from the more usual WKBJ approximation (Budden, 1961, Ginsberg,
1961) in offering a solution consistent with the full-wave problem but requiring that the
perturbations to the profile are small. The WKBJ approach, by comparison, ignores the
finite wavelength but can be applied to the zeroth order problem, and hence to large
perturbations, at least in principle.

The present approach parallels more recent calculations by Mazzucato and Nazikian
(1991), Garcia et al (1989) and Zou et al (1990) but goes further in presenting more
systematic numerical results and in addressing the matter of the wave attenuation, which
proves to have some potentially very important consequences, and correlations.

It must be emphasized that it is far from clear whether even this extended treatment
of the problem can adequately describe the physics of actual experiments. It may be that
intrinsically multi-dimensional effects, such as Bragg reflection from rippled surfaces, are
predominant in the experiments (Irby, 1990). Nevertheless, it seems essential to conduct
this more thorough analysis of the one-dimensional problem so as to understand what can
and cannot be explained on the basis of a one-dimensional approach.

2. One-dimensional full-wave reflectometry

We consider a plasma slab in which all gradients are perpendicular to the magnetic field,
$B$, which is in the $z$-direction. We suppose the wave under analysis to propagate in the
direction of the gradients, which we take as the $x$-direction. In the context of a full-wave
analysis, this means that the only non-zero derivatives in the problem are $\partial/\partial x$. This is
thus a one-dimensional analysis.

We further suppose that the plasma can be described by a local response in the form
of a dielectric tensor $\varepsilon$ that is a function only of the frequency, $\omega$, and position, $x$, so that
the wave equation is

$$\nabla \times (\nabla \times E) - \frac{\omega^2}{c^2} \varepsilon \cdot E = 0. \quad (1)$$

Finally, we suppose that the cross terms $\varepsilon_{xz}$ and $\varepsilon_{yz}$ are zero. These criteria are satisfied
by the cold plasma description that is normally used for this type of wave analysis. Under these assumptions, it may readily be shown that the wave equation (1) separates into two uncoupled scalar wave equations:

\[
\left[ \frac{d^2}{dz^2} + \frac{\omega_p^2}{c^2} \epsilon_{zz} \right] E_z = 0,
\]

(2)

for the ordinary wave, and

\[
\left[ \frac{d^2}{dz^2} + \frac{\omega_p^2}{c^2} (\epsilon_{yy} - \epsilon_{yz} \epsilon_{zy} / \epsilon_{zz}) \right] E_y = 0,
\]

(3)

for the extraordinary wave. The cold plasma values of the relevant dielectric tensor elements, neglecting the ion response, are

\[
\begin{align*}
\epsilon_{zz} &= \epsilon_{yy} = 1 - \omega_p^2 / (\omega^2 - \Omega^2) \\
\epsilon_{xy} &= -\epsilon_{yz} = i \omega_p^2 \Omega / \omega (\omega^2 - \Omega^2) \\
\epsilon_{xx} &= \omega_p^2 / \omega^2,
\end{align*}
\]

(4)

where \( \omega_p \) and \( \Omega \) are the electron plasma and cyclotron frequencies respectively.

Equations (2) and (3) are both of the form

\[
\frac{d^2 E}{dz^2} + k^2 E = 0,
\]

(5)

i.e. of the Helmholtz type but with the parameter \( k^2 \) a function of \( z \). For the ordinary wave

\[
k^2 = (\omega^2 - \omega_p^2) / c^2 = k_0^2 (1 - n / n_c),
\]

(6)

where \( k_0 = \omega / c \) and \( n_c \) is the critical density, \( \omega^2 \epsilon_0 m_e / e^2 \). While for the extraordinary wave

\[
k^2 = k_0^2 \left[ 1 - \frac{\omega_p^2 (\omega^2 - \omega_p^2)}{\omega^2 (\omega^2 - \Omega^2 - \omega_p^2)} \right].
\]

(7)
The more general situation where the wave is given an additional variation \( \exp(ik_yy) \) also gives rise to uncoupled scalar wave equations (see e.g. Budden, 1961) and its formalism for reflectometry has been set forth by Mazzucato and Nazikian (1991). For the ordinary wave the modification is trivial, subtracting \( k_y^2 \) from the expression for \( k^2 \) (Eq(6)). However, for the extraordinary wave a more complex equation arises, possessing first derivative terms. Such an equation can be recast into the form of Eq(5) but only by a transformation of the independent variable, \( x \). Since discussion of this process is beyond the present scope, one may consider the analysis as representing the extraordinary mode only at normal incidence.

The general situation in reflectometry is illustrated in Fig 1(a). The wave number, \( k^2 \), has a functional form such that for large positive \( x \), deep inside the plasma, \( k^2 \) becomes large and negative, the wave is cut off. For increasingly negative \( x \), there is some value \( x_e \), equivalent to the plasma edge, beyond which \( k^2 \) is constant and positive. In the intervening region \( k^2 \) varies continuously.

The second order linear differential equation (5) has two independent solutions, which we will denote \( \psi_1 \) and \( \psi_2 \). The physically significant solution is, of course, the one that tends to zero for large \( x \). Let us suppose that \( \psi_1 \) is this solution. It is shown in Fig 1(b) for our illustrative profile.

We wish to understand the sensitivity of the reflectometer signal to influence from different positions. To discuss this we consider a small perturbation to the \( k^2 \) profile, arising, for example, from a density perturbation. Specifically, we will reconstruct the perturbed solution using the Green's function, which is the solution of the problem

\[
\frac{d^2G}{dx^2} + k^2G = \delta(x - \xi), \tag{8}
\]

under the boundary conditions \( \psi = 0 \) for as \( x \to \infty \) and \( U(\psi) = 0 \), where \( U(\psi) \) is a linear combination of \( \psi \) and \( d\psi/dx \) evaluated at some reference point, \( x_0 \) say, in the vacuum region, \( x_0 < x_e \). We shall discuss in a moment the required form of \( U \).
The Green's function (see e.g. Stakgold, 1979) may easily be shown to be

\[
G(x, \xi) = \psi_1(\xi)\psi_2(x) - \frac{U(\psi_2)}{U(\psi_1)}\psi_1(x)\frac{1}{W(\xi)}, \quad \text{for } x < \xi,
\]

\[
= \psi_1(x)\psi_2(\xi) - \frac{U(\psi_2)}{U(\psi_1)}\psi_1(\xi)\frac{1}{W(\xi)}, \quad \text{for } \xi < x,
\]

(9)

where \(W(\xi)\) is the Wronskian,

\[
W(\xi) = \psi_1(\xi)\psi_2'(\xi) - \psi_1'(\xi)\psi_2(\xi).
\]

(10)

and primes denote differentiation by the argument. Then the solution of the general inhomogeneous problem

\[
\frac{d^2\psi}{dx^2} + k^2\psi = f(x),
\]

(11)

with the boundary conditions \(\psi(\infty) = 0, U(\psi) = 0\) is

\[
\psi(x) = \int_{\xi_0}^{\infty} G(x, \xi)f(\xi)d\xi.
\]

(12)

Proceeding in a manner that amounts to the Born approximation, we seek a solution to the perturbed equation

\[
\frac{d^2E}{dx^2} + [k^2(x) + \tilde{k}^2(x)]E = 0,
\]

(13)

by assuming that \(\tilde{k}^2\) is small compared to \(k^2\) so that the solution may be obtained approximately in the form of an expansion \(E_0 + \tilde{E}\) with \(E_0\) the solution of the original equation (5) and

\[
\frac{d^2\tilde{E}}{dx^2} + k^2\tilde{E} = -k^2 E_0.
\]

(14)

This first approximation will be good provided \(\tilde{E} \ll E_0\) at all \(x\). Putting \(E_0 = \psi_1\) as the zeroth order solution, substituting \(f = -\tilde{k}^2\psi_1\), and using Eq(12) for a point in the vacuum region, so that only the case \(x < \xi\) applies, we get

\[
\tilde{E}(x) = -[\psi_2(x) - \frac{U(\psi_2)}{U(\psi_1)}\psi_1(x)]\int \tilde{k}^2(\xi)\frac{[\psi_1(\xi)]^2}{W}d\xi.
\]

(15)
Thus the first order perturbation to the solution is a mixture of the two independent solutions to the unperturbed problem, with coefficients given by eq(15).

In one-dimensional reflectometry for a lossless medium, $k^2$ is real. However, it is of considerable interest to consider cases where $k^2$, and the solutions to the equation (5), are complex. Such a case describes a situation in which either the medium is absorbing or else $\omega$ has an imaginary component owing to a temporal variation in amplitude. More important, perhaps, this case can be used to construct a one-dimensional model of the multi-dimensional physical situation when the waves are somewhat diverging. As a consequence of such divergence, the wave amplitude along the central path of the ray will experience attenuation due to the spreading of the antenna pattern. To describe this situation rigorously would require a multi-dimensional treatment, in which the transverse wave pattern was described as an appropriate spectrum over, for example, $k_y$. Each component of the spectrum would then be solved using the present approach, and the resulting electric field reconstructed as the (coherent) sum over the $k_y$-spectrum. Such a treatment is beyond the scope of the present work, since the physically important case with non-zero value of $k_y$ leads to mode coupling, and the scalar wave equation fails. However, it is proposed here that the key physical effects of the transverse wave spreading can reasonably be modelled by the addition of an imaginary part to $k^2$. This imaginary part is regarded as a parameter that can be adjusted to model approximately the effect of a specific antenna configuration.

The analysis so far is fully applicable to cases where $k^2$ is complex. Moreover the use of complex representations provides the most convenient way of specifying the appropriate boundary conditions and deducing the reflectometry signal which corresponds to the solution, Eq(15).

In the vacuum region, where $k_v$ is constant, the unperturbed solution, $\psi_1$, can be decomposed into forward and backward propagating waves:

$$\psi_1 = A_1 \exp(ik_v(x - x_0)) + B_1 \exp(-ik_v(x - x_0)),$$

(16)
where an additional factor $\exp(-i\omega t)$ is understood and $A_1$ and $B_1$ are the complex amplitudes. Then at the position $x_0$, which we consider to be our reference position, one can readily show that

$$A_1 = (\psi_1 - i\psi'_1/k_v)/2; \quad B_1 = (\psi_1 + i\psi'_1/k_v)/2.$$ (17)

The second solution, $\psi_2$, can also be described in terms of such amplitudes, $A_2$ and $B_2$, which we are free to choose as we like, so long as the result is linearly independent of $\psi_1$. For definiteness we make the choice:

$$A_2 = -iA_1 \quad \text{and} \quad B_2 = iB_1.$$ (18)

The Wronskian at the point $\xi = x_0$ is then

$$W = (A_1 + B_1)ik_v(A_2 - B_2) - (A_2 + B_2)ik_v(A_1 - B_1)$$
$$= 4k_vA_1B_1 = k_v(\psi_1^2 + \psi_1'^2/k_v^2).$$ (19)

It follows from general Wronskian theory (Stakgold, 1979) that for an equation such as Eq(5), having no first order term, the Wronskian, $W$, is independent of $\xi$. For the present choice, then, the value of $W$ given by Eq(19), evaluated at $x_0$, applies for all $\xi$.

We shall assume that the physical situation to be described is one in which the launched (forward) wave amplitude is determined by the apparatus used and is fixed. Thus the boundary condition to be applied at $x_0$ is that the forward amplitude of the perturbation should be zero. This requires us to take

$$U(\psi) = \frac{1}{2}(\psi - \frac{i}{k_v}\psi')|_{x_0}.$$ (20)

With this choice, one gets $U(\psi_2)/U(\psi_1) = A_2/A_1$, and consequently the perturbation gives rise to forward amplitude, $\bar{A} = 0$ (by construction), and backward amplitude

$$\bar{B} = -[B_2 - B_1A_2/A_1] \int k^2 \frac{\psi_1^2}{W} d\xi = -2iB_1 \int k^2 \frac{\psi_1^2}{W} d\xi.$$ (21)
This expression gives the general form of the perturbation to the solution, and may under some circumstances be the most useful form. However, particularly for one-dimensional reflectometry, the key quantity is usually the phase difference of the reflected wave. This may be deduced immediately from Eq(21). The total reflected wave is \( B_1 + \widetilde{B} \), whose phase angle relative to \( B_1 \) is

\[
\tilde{\phi} = -2\Re \left( \int \frac{k^2 \psi^2}{W} d\xi \right),
\]

provided \( \tilde{\phi} \) is small. [\( \Re(f) \) denotes real part of \( f \).]

Our result, Eq(22), amounts to a demonstration that for perturbations of the system resulting in a phase perturbation that is everywhere small, the phase perturbation outside the plasma is given by a simple integral over the whole plasma of the perturbation to \( k^2 \) times the square of the solution of the unperturbed system, divided by the (constant) Wronskian, \( W \).

For comparison, it may be noted that the usual WKBJ approximation, \( \phi = -2 \int k d\xi + \pi/2 \), gives rise to a linearized phase perturbation expression,

\[
\tilde{\phi} = -2 \int \frac{k^2}{2k} \frac{1}{2k} d\xi,
\]

(23)

Our full-wave result shows that this WKBJ estimate of the phase shift is incorrect in not accounting for the modulation of the reflectometer sensitivity proportional to \( \psi^2 \). What is more, our result shows how properly to account for the region where \( k^2 \) passes through zero. The full-wave treatment avoids the unphysical divergence in the sensitivity that occurs in the WKBJ approximation.

Although we have calculated only the linearized phase sensitivity in the vicinity of a zeroth-order solution, this linearized result is what is required to determine the sensitivity, to local variations of \( k^2 \), of any reconstruction of the \( k^2 \)-profile. In other words, if we have some presumed profile, which would give rise to the measured phase shift, then the amount by which the measured phase is altered by small changes in the presumed profile
determines the sensitivity of the phase as a measurement of the profile. The spatial form of this sensitivity is our result: $\mathcal{R}(\psi_1^2/W)$. This is illustrated in Fig 1(c) (for which all quantities are real since $k^2$ is real for this example). Its characteristics are general, independent of the particular (real) $k^2$ profile adopted: approximately sinusoidal variation with a substantial swelling of amplitude and wavelength where $k^2$ passes through zero, followed by an exponentiation to zero. The WKBJ result, i.e. $1/2k$, is over-plotted on the same scale for comparison. As expected, it gives the average of the full-wave result in the oscillatory region. Near the cut-off point, $k = 0$, however, the singularity leads to rather appreciable deviation. This comparison is helpful, also, because the validity of Eq(23) as a linearization of the WKBJ solution depends on the criterion $\frac{k^2}{\hat{r}} << k^2$, which is less restrictive than the criterion, $\hat{r} << 1$, for the first Born approximation to be accurate. Those aspects of our results that arise from the average behaviour of $\mathcal{R}(\psi_1^2/W)$, regardless of its oscillatory character, are therefore still valid under the less restrictive criterion.

The linearized response also allows us to calculate immediately the group delay, $d\phi/d\omega$. This is the quantity that is measured by an experiment that uses amplitude or frequency modulation so as to avoid the ambiguities of the phase delay, $\phi$ (e.g. Doane, 1981). The group delay gives the phase shift of the modulation, and is of course equivalent to a measurement of the round-trip time travelling at the group velocity, $d\omega/dk$. It is also the form that appears in the usual (Abel) inversion expression for ordinary-wave reflectometry under the WKBJ approximation (see e.g. Hutchinson, 1987). Since $k^2$ is a function of $\omega$ as well as $x$, Eq(22) gives immediately:

$$\frac{d\phi}{d\omega} = -2\mathcal{R}\left(\int \frac{\partial k^2 \psi_1^2}{\partial \omega W} d\xi\right) = -2\mathcal{R}\left(\int \frac{\partial k}{\partial \omega} 2k \frac{\psi_1^2}{W} d\xi\right). \quad (24)$$

For the ordinary wave, since by Eq(6) $\partial k^2/\partial \omega = 2\omega/c^2$, independent of the plasma parameters, the expression for the group delay is particularly simple:

$$\frac{d\phi}{d\omega} = -\frac{4\omega}{c^2} \int \mathcal{R}\left(\frac{\psi_1^2}{W}\right) d\xi, \quad (25)$$

proportional to a straightforward integral of $\psi_1^2/W$. 

10
3. General Characteristics

The formal results of the previous section need to be related to an experiment by introducing the actual form of $k^2$ and finding the solution $\psi_1$. This can, and probably often should, be done numerically. However, it is useful for the purpose of a more general analysis to examine an approximate solution of the unperturbed differential equation so as to provide the typical scale lengths of the problem, for example the width of the rightmost lobe of the function $\psi_1^2$.

This approximate general solution is provided by using the WKBJ approximation (in the region $x < x_j$, say) away from the reflection point, $x_c$ where $k^2 = 0$; and approximating the $k^2$ profile as linear in the region of the reflection point $(x > x_j)$, so that the solution there is the Airy integral function, $Ai$ (Abramowitz and Stegun, 1965). This is precisely the approximate solution that is developed in detail by Ginsberg (1961) in obtaining the usual expression for the phase in the WKBJ approximation. The solution can then be written

$$\psi_1 = \left(\frac{k_0}{k}\right)^{\frac{1}{2}} \sin \left(\int_x^{x_c} k \, dx + \frac{\pi}{4}\right) \text{ for } x < x_j \text{ and }$$

$$\psi_1 = \frac{\left(\pi k_0\right)^{\frac{1}{2}}}{|k^2|^\frac{1}{4}} Ai \left(|k^2|^\frac{1}{4}[x - x_c]\right) \text{ for } x \geq x_j.$$  \hspace{1cm} (26)

Again, we are writing $k^2$ for $\partial k^2/\partial x$, which should be understood as being evaluated at the reflection point, $x_c$. The choice of amplitude coefficients given by Eq(18) was made specifically so that the second solution, $\psi_2$, is identical in form to Eq(26) except that $Ai$ must be replaced by the second Airy function, $Bi$, and sine by cosine.

The position $x_j$ where the the WKBJ solution is joined to the Airy function by matching to its asymptotic expansion needs to be far enough from $x_c$ to make the asymptotic expansion accurate, but close enough for the straight-line approximation to $k^2$ to be accurate. Ginsberg (1961) notes that if the argument of the Airy function, $|k^2|^{1/3}(x - x_c)$, is less than -5 the error in the asymptotic expansion is less than 1%. This argument corresponds roughly to the position of the penultimate positive maximum of $Ai$. Provided the
\( k^2 \) curve is relatively straight as far from \( x_c \) as this, the approximate solution will be quite accurate.

For reference, it may be noted that \( \text{Ai}(x) \) has zeros at \( x = -2.34, -4.09, \ldots \) and \( \text{Ai}' \) has zeros at \( x = -1.02, -3.25, -4.82, \ldots \) where \( \text{Ai} \) has the values 0.5357, -0.4190, and 0.3804 respectively. The half maximum points of the last lobe of \( \text{Ai}^2 \) are at \( x = -0.092 \) and -1.722, giving a full width of 1.63.

The scale factor, \( |k^2|^{1/2} \), relating physical position to Airy function argument, is particularly simple for the ordinary mode, described by Eq.(6). We find

\[
|k^2|^{1/2} = |k_n^2 n'/n_c|^{1/2} = (k_n^2 / L_n)^{1/2},
\]

where \( L_n \) is the density scale length. Thus, for example, the full--width--half--maximum in physical space of the last lobe of \( \psi^2_1 \) is 1.63\( L_n / (k_n L_n)^{1/2} \), intermediate between the density scale length and the inverse vacuum wavenumber \( k_n^{-1} \).

One must be wary of implying that the width of the last lobe represents the actual spatial resolution of the measurement. It does represent in some circumstances approximately the minimum resolution. However, there can be substantial contribution to the integral, Eq(22), from the oscillatory region. Moreover, the larger the value of \( k_n L_n \), the larger the fraction of the integral that comes from the oscillatory region. This is illustrated in Fig 2 where we show the integral \( \int_{-\infty}^{\infty} \text{Ai}^2/W dz' \) as a function of \( x \). This integral would give the phase shift for a perturbation consisting of a uniform increase in \( k^2 \) or equivalently a uniform \( x \)-shift of the \( k^2 \) profile. It is also what is required for evaluating the group delay in ordinary mode reflectometry. The graph can be interpreted as representing a whole range of possible plasmas having linear \( k^2 \) profiles. One views the abscissa as the position of the plasma edge relative to the cut-off point, in scaled units. The last lobe of the \( \text{Ai}^2 \) function contributes about 1.54 to the integral. Thus, the fraction that comes from the last lobe depends on the distance to the edge measured in scaled units, \(-x\), which is \((k_n L_n)^{1/2}\) for the ordinary mode. Also shown in Fig. 2 is the corresponding phase shift estimate using
the WKBJ approximation, which proves to be equal simply to $k$. It differs very little from
the full-wave result when integrated in this manner, particularly in the region where $k$ is
large. This fact confirms that one can estimate the contribution from this region using
the simple WKBJ estimate $\int (1/2k)dx$ with negligible error provided that the $k^2$ profile is
much broader than the wavelength of the solution Eq(26).

Clearly, from measurements of the perturbed phase at a single reflectometry wave
frequency, $\omega$, it is impossible to deduce the profile of a general perturbation $k^2$. On the
other hand, if we had values of $\tilde{\phi}$ for all relevant values of $\omega$ then it is formally possible to
invert the integral equation

$$\tilde{\phi}(\omega) = -2 \int k^2(\xi, \omega) R \left( \frac{\psi^2_1(\xi, \omega)}{W(\omega)} \right) d\xi. \quad (28)$$

This may be understood by thinking of the function $R(\psi^2_1(\omega, \xi)/W(\omega))$ as a kernel asso-
ciated with a type of integral transform, in the same way as $\exp(i\omega t)$ is the kernel of the
Fourier transform, or $r/(r^2 - x^2)^{1/2}$ is the kernel of the Abel transform. Under certain
general conditions such transforms can be inverted. And indeed, if one introduces a regu-
larization cost functional that must be minimized, then the inversion can be made unique.
This amounts to saying that given $\tilde{\phi}(\omega)$, for all relevant frequencies there is a unique fitted
solution, $k^2(\xi)$, that satisfies Eq(28) and also minimizes the chosen functional. In practice,
if the kernel has some minimum structure size, then the resolution of the solution will be
limited to that minimum size, especially in the presence of uncertainties in the measure-
ments. We have shown that for our kernel the minimum size, i.e. the width of the last
lobe of $\psi^2_1$, is about $1.63/|k^2|^\frac{1}{2}$, evaluated at the cut-off point, which is $1.63L_n/(k_nL_n)^\frac{3}{2}$
for the ordinary mode.

When we allow $k^2$ to have an imaginary part due to absorption or antenna-pattern
divergence, the solutions change as shown in Fig 3. For illustrative purposes Fig 3 shows
cases in which the $k^2$ profile is $k^2 = -x + \Im(k^2)$ with constant imaginary part, $\Im(k^2)$. Fo-
cussing on $R(\psi^2_1/W)$, we see that it takes on an increasingly bipolar oscillatory structure
as the imaginary part of $k^2$ is increased. This is most noticeable at the most negative $x$-values where the attenuation has most effect in reducing the reflected wave. The physical reason why the actual amplitude of $R(\psi_1^2/W)$ increases at the edge with increasing imaginary part is that the unperturbed reflected wave amplitude, $B_1$, is reduced. Therefore the perturbed wave arising from the (back-scattering of the) forward wave (which is itself much less attenuated than the backward wave) has a larger amplitude relative to $B_1$. It therefore can perturb the phase of the backward wave more.

This observation holds only for the envelope of the oscillation, however. When one integrates over these oscillations to obtain $\int \psi_1^2/W \, dx$ one finds that the result is almost the same as given in Fig 2. The only difference is a slightly enhanced ripple. The curve otherwise falls right on top of the one shown there. Thus, for perturbations that have a broader profile than the wavelength of the oscillations the effects of wave attenuation are not very important.

4. Correlation Reflectometry

Based on the foregoing relations, we are now in a position to give explicit expressions for the quantities measured by correlation reflectometry. We shall assume that the output of each reflectometer channel is proportional to the phase shift, $\phi$. A simple homodyne reflectometer, however, gives a signal that is proportional not to $\phi$ but to $\sin(\phi_0)\phi$. That is, its sensitivity is modulated according to the zeroth order phase shift. This effect alone could be responsible for observation of low correlation between reflectometry channels at different frequencies; since variations in $\phi_0$, different in different channels, due to slow changes in the average density profile, could rapidly average the correlation to a small value. For this reason it seems essential, if unambiguous results are to be obtained in correlation reflectometry, that some form of quadrature system should be used so as to measure $\phi$ properly. (This effect was discussed briefly by Hanson et al, 1990, who have implemented a quadrature system. However, the correlations they formed were not exactly
the phase correlations.

We shall consider, then, two reflectometer signals, $\tilde{\phi}_a$ and $\tilde{\phi}_b$ obtained from the same plasma path but with different frequencies, $\omega_a$ and $\omega_b$. Each of them can be expressed as an integral of the form of Eq.(22). We shall drop the subscript 1 from the solution of the unperturbed equation in this section and use subscripts $a$ and $b$ to refer to the two channels. Also, for notational convenience we shall write

$$K(\xi, \omega) \equiv R\left(\frac{\psi^2(\xi, \omega)}{W(\omega)}\right)$$

(29)
as shorthand for the kernel function. The product of the two reflectometer signals is then

$$\tilde{\phi}_a \tilde{\phi}_b = 2 \int k_2^2(\xi_a)K(\xi_a, \omega_a)d\xi_a 2 \int k_2^2(\xi_b)K(\xi_b, \omega_b)d\xi_b$$

$$= 4 \int \tilde{k}_2^2(\xi_a)\tilde{k}_2^2(\xi_b)K_a K_b d\xi_a d\xi_b.$$  

(30)

When we take the ensemble average of this equation so as to obtain the quantity $<\tilde{\phi}_a \tilde{\phi}_b>$ we obtain an identical double integral of the quantity $<\tilde{k}_2^2(\xi_a)\tilde{k}_2^2(\xi_b)>$. If we know the unperturbed profiles, then $\psi^2/W$ may be calculated for any $\omega$, specifically $\omega_a$ and $\omega_b$. Therefore the integral relation between the signal correlation and the perturbed wave-number is fully defined.

For the rest of our discussion we shall consider only the situation in which the square wave-number perturbation is independent of the wave frequency $\omega$; so that $\tilde{k}_2^2(\xi) = \tilde{k}_2^2(\xi) = \tilde{k}^2(\xi)$. This is in fact the case for the ordinary wave, since

$$\tilde{k}^2 = -\tilde{\omega}^2/c^2 = -(e^2/\epsilon_0 m_e c^2) \tilde{n}_e.$$  

(31)
The correlation signal is then a weighted integral of the spatial correlation function of the square wave-number perturbation:

$$C_{k2}(\xi_a, \xi_b) \equiv <\tilde{k}_2^2(\xi_a)\tilde{k}_2^2(\xi_b)> = (e^2/\epsilon_0 m_e c^2)^2 <\tilde{n}_e(\xi_a)\tilde{n}_e(\xi_b)>.$$  

(32)
The general expression for the correlation coefficient is

$$\rho_\phi(\omega_a, \omega_b) \equiv \frac{<\tilde{\phi}(\omega_a)\tilde{\phi}(\omega_b)>}{[<\tilde{\phi}^2(\omega_a)><\tilde{\phi}^2(\omega_b)>]^{1/2}},$$

(33)

15
where, in our case,

$$\langle \tilde{\phi}(\omega_a)\tilde{\phi}(\omega_b) \rangle = 4 \int \langle \tilde{k}^2(\xi_a)\tilde{k}^2(\xi_b) \rangle K(\xi_a,\omega_a)K(\xi_b,\omega_b)d\xi_a \, d\xi_b ,$$

(34)

and

$$\langle \tilde{\phi}^2(\omega_a) \rangle = 4 \int \langle \tilde{k}^2(\xi_a)\tilde{k}^2(\xi_b) \rangle K(\xi_a,\omega_a)K(\xi_b,\omega_b)d\xi_a \, d\xi_b .$$

(35)

The correlation function $\langle \tilde{k}^2(\xi_a)\tilde{k}^2(\xi_b) \rangle$ contains, of course, the statistical information (up to second order) of the random variable $\tilde{k}^2$. This is therefore what we would like to deduce. Just as for the inversion described in section 3, there is plainly no one-to-one relationship that would allow us to deduce a value of the correlation function of $\tilde{k}^2$ from a single measurement of the reflectometer correlation at two given frequencies. On the other hand, if we had values of $\langle \tilde{\phi}_a\tilde{\phi}_b \rangle$ for all relevant values of $\omega_a$ and $\omega_b$ then it is formally possible to invert the double integral equation (30). That is, there exists a unique fitted solution, $\langle \tilde{k}^2(\xi_a)\tilde{k}^2(\xi_b) \rangle$, that satisfies Eq(30) and also minimizes some chosen functional.

Rather than develop further formal solutions to the inverse problem, which would be of doubtful practical value, it seems more useful to give some systematic examples of the forward problem. That is, we shall choose some illustrative forms of the correlation function $\langle \tilde{k}^2(\xi_a)\tilde{k}^2(\xi_b) \rangle$ and calculate the phase correlation functions that they would give. For this purpose we shall adopt an unperturbed $k^2$ profile that is computationally convenient and serves to illustrate the general conclusions we wish to draw. We choose, as before, a linear $k^2$ profile whose slope is independent of frequency. This represents the physical situation of a linear density profile for ordinary mode reflectometry. The chief convenience is that the unperturbed solution is then given throughout the plasma region (which is the only region that matters) by the Airy function solution of Eq(26) with the scale factor $|k|^{2 \prime}$ a constant independent of space and frequency. The parameters of the solution that are functions of frequency are then just $k_0 = \omega/c$ and $\pi_c$. When we form
\( \psi^2/W \) even the \( k \) variation cancels out and if we scale the \( x \)-coordinate to make \( |k^2| = 1 \) we obtain

\[
K(x, \omega) \equiv R(\psi^2(x, \omega)/W) = \pi \text{Ai}^2(x - x_c(\omega)) .
\] (36)

5. Turbulent Fluctuations

We consider situations in which \( \tilde{k}^2 \) has a uniform spatial profile, by which we mean that it is a random variable that is stationary in the sense that \( \langle \tilde{k}^2(\xi_a)\tilde{k}^2(\xi_b) \rangle \) is a function only of the difference \( \xi_a - \xi_b \). We may then write

\[
\langle \tilde{k}^2(\xi_a)\tilde{k}^2(\xi_b) \rangle = C_{k^2}(\xi_a - \xi_b) ,
\] (37)

and the only choice we then have to discuss is what is the form of the correlation function \( C_{k^2}(\xi) \).

Two limits of long and short correlation-length are particularly simple. When \( C_{k^2}(\xi) \) is so wide that it can be considered a constant, the limit of long correlation length, then

\[
\langle \tilde{\phi}(\omega_a)\tilde{\phi}(\omega_b) \rangle = C_{k^2} 4 \int K(\xi_a, \omega_a) d\xi_a \int K(\xi_a, \omega_a) d\xi_b ,
\] (38)

and

\[
\rho_{\phi}(\omega_a, \omega_b) = 1 .
\] (39)

The form of the integrals in Eq(38) have already been given in Fig 2.

On the other hand, when the correlation function is highly localized, so that \( C(\xi) \) is significant only for very small values of \( \xi \) then we may approximate it as a delta function and deduce immediately that

\[
\langle \tilde{\phi}(\omega_a)\tilde{\phi}(\omega_b) \rangle = 4 \int K(\xi_a, \omega_a) K(\xi_a, \omega_b) d\xi .
\] (40)

Substituting the squared Airy function for \( K \) we can then obtain the correlation coefficient as

\[
\rho_{\phi}(\omega_a, \omega_b) = \frac{\int \text{Ai}^4(x - x_c(\omega_a))\text{Ai}^2(x - x_c(\omega_b)) dx}{\left[ \int \text{Ai}^4(x - x_c(\omega_a)) dx \int \text{Ai}^4(x - x_c(\omega_b)) dx \right]^{1/2}} .
\] (41)
More general possible forms of $C_{k2}$ are of infinite variety. However, as an example of variable correlation length we may take a Gaussian shape in the scaled dimension, $\xi$, with width $\sigma$

$$C_{k2}(x) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(\frac{-x^2}{2\sigma^2}\right).$$

(42)

This corresponds to a Gaussian fluctuation power spectrum of unit height, and width $1/\sigma$. The resulting phase correlation functions and coefficients can be obtained by performing the integrals in Eqs(34, 35).

For purposes of relating the results to physical quantities, it may be noted that in ordinary-wave reflectometry this correlation function corresponds to a mean-square density fluctuation level

$$<\bar{n}_e^2> = \frac{|k_2|^4/3}{(2\pi)^{1/2}(4\pi r_e)^2\sigma},$$

(43)

where $r_e$ is the classical electron radius $e^2/(4\pi\varepsilon_0 m_e c^2) = 2.818 \times 10^{-15}$ m. Values of the phase correlation function for a given density fluctuation can be obtained using this scaling factor. However, the density (fluctuation) we are discussing is the mean across the antenna pattern, $\int n_e dydz/\int dydz$, which may have a substantially smaller fluctuation than the local density. The width $\sigma$ is in scaled dimensionless units; so that the physical width is $\sigma/|k_2|^\frac{2}{3}$.

The extent of the integrations in Eqs(34, 35) is, by implication, only up to the edge of the plasma, $x_e$. We may choose to measure distances from the cut-off position of one of the waves and to regard the correlation function (Eq(34)) and the correlation coefficient as functions of the distance of the plasma edge from the cut-off position in scaled units. The results are then universal for these ‘stationary’ profiles.

Figs. 4 and 5 show some examples. For various fixed values of the $x$-shift distance, $x_{ca} - x_{cb}$, between the cut-off positions of the two frequencies, we plot, as a function of $x_e$, the integrals from position $x_e$ to infinity, where $x_e$ is measured relative to the cut-off position $x_{ca}$. We thus obtain universal curves that allow us not only to obtain the
observable correlation functions for a range of different plasmas, but also to deduce where in space the contribution to the correlation function comes from. This second, and very important, factor is given by realizing that the contribution arising from plasma between any two positions \(x_1\) and \(x_2\) to the correlation function for a plasma whose edge, \(x_e\), is further from the cut-off position, \(x_{ce} (= 0)\), than \(x_1\) and \(x_2\) is simply the difference between the values of the correlation function evaluated at \(x_e = x_1\) and \(x_e = x_2\).

As a summary, we also show in Fig. 6 plots of the correlation coefficient evaluated at \(x = 32.5\) (the left hand edge) versus the \(x\)-shift, \(x_{ce} - x_{ek}\), for various values of the width, \(\sigma\), which is the correlation length of the fluctuations.

For this situation, in which \(k^2\) is real, we see that the correlation function can be thought of as having two main types of contribution. First, a step arising from the last lobe of \(\psi^2(\omega_k)\), the lower-frequency wave-function. Second, an integrated contribution from the oscillatory regions of both wave-functions. The relative importance of these two contributions depends on the relative distance to the plasma edge (i.e. \(x_e\)) but for typical values of \(x_e\) between 20 and 40, the percentages are very roughly 50-50. The height of the step is strongly affected by the correlation length of \(k^2\) whereas the oscillatory region contribution is much less affected. As \(\sigma\) increases this latter contribution experiences smoothing of the ‘beat’ pattern between the two oscillating sensitivities, most noticeable for zero \(\sigma\), and a depression of the correlation function for small \(x\)-shift (which can also be thought of as a removal of the beats).

The consequence, shown in Fig. 6, is that the correlation coefficient of the reflectometer signals as a function of \(x\)-shift bears only a distant relationship to that of the \(k^2\) fluctuations. In particular, the reflectometer correlation coefficient remains large regardless of how short the fluctuation correlation length is: for example roughly 0.5 at an \(x\)-shift of 10 for \(\sigma s\) in the range 0 to 3. Thus the observation of substantial correlation at large shifts cannot safely be interpreted as indicating fluctuations with comparable correlation lengths.
Conversely, the observation of negligibly small correlations (e.g. Costley et al, 1990, Hanson, et al, 1990) is difficult to reconcile with any values of the x-shift and fluctuation correlation length, on the basis of one-dimensional reflectometry with real \( k^2 \). It is therefore of some interest to examine situations with an imaginary component to \( k^2 \). We find that the results can be qualitatively altered.

We consider the case \( I(k^2) = 0.3 \) (in scaled units). This has the effect of removing most of the swelling of \( R(\psi^2/W) \), and most importantly making, \( R(\psi^2/W) \) mostly 'bipolar' in the region of negative \( x \). (See Fig 3.) The consequence for the correlation function is shown in Figs. 7, 8 and 9. The auto-correlation (i.e. the correlation function for zero x-shift) gains an enhanced contribution from the oscillatory region for \( \sigma = 0 \). However, this contribution is removed when the fluctuation correlation length is increased to be greater than the wavelength of the oscillations of \( R(\psi^2/W) \). The function then reverts to that for real \( k^2 \). This has important consequences for the correlation coefficient, causing it to decrease substantially when \( \sigma \) is short.

Thus we may conclude that provided there are negligible fluctuations with correlation lengths short compared with the oscillation wavelength (i.e. roughly half the vacuum wavelength at a minimum) the effects on one-dimensional correlation reflectometry of attenuation of the wave amplitude due to absorption or, more likely, beam divergence, is not important. However, in the very likely event that there are significant fluctuations with short wavelength, especially at the edge of the plasma, the auto-correlation can be dominated by them, and the correlation coefficient can be greatly depressed. In this situation the observation of low correlation coefficient can be regarded as a sign of short-correlation-length fluctuations but not near the cut-off position, where the bipolar nature of \( R(\psi^2/W) \) is least; more likely nearer the antenna, i.e. at the plasma edge.
6. Coherent Fluctuations

The characteristics described in the previous section seem to apply to virtually all cases in which the correlation coefficient of the $\overline{k^2}$ fluctuations is monotonically decreasing. However, when one is interested in coherent sinusoidal waves, rather than highly incoherent fluctuations, the characteristics prove to be different. The coherent case can generically be described by a correlation coefficient

$$C_{k^2}(\xi) = \exp(-\xi^2/2\sigma^2) \cos(k_f \xi).$$

(44)

Thus $\sigma$ is the overall width, but is the envelope of a wave with dominant wave-number $k_f$. Such a correlation arises, of course, from a fluctuation wave-number power spectrum of Gaussian shape, width $1/\sigma$, centered at $k_f$. In this case, we choose to normalize $C_{k^2}(0)$ to unity, which gives a power spectrum of fixed area (rather than height) and mean-square fluctuation $\bar{n}_k^2 = |k^2|^{4/3}/(4\pi r_s)^2$ independent of $\sigma$.

The reflectometry correlation functions and coefficients that arise from such fluctuations have been evaluated. Examples are shown in Fig. 10. The dominant contribution to the correlation function arises from the position at which the oscillations of the fluctuation match those of the weighting function $R(\psi^2/W)$. This occurs where $2k = k_f$, which is naturally the Bragg condition for backscattering. This position is close to the cut-off point of the waves if $k_f$ is small enough. Specifically, the main contribution comes mostly from the last lobe of the unperturbed wave-function if $k_f \lesssim 3$ in scaled units, i.e. $k_f/|k^2|^{1/3} \lesssim 3$ in physical units. [The criterion given by Mazzucato and Nazikian, viz. $k_f \lesssim k_*/2.5$, is not general, but applies only to their specific profile and characteristic length].

A remarkable transformation of the reflectometry correlation coefficient takes place as we increase the product $k_f\sigma$, which is basically the number of oscillations in the total width of the fluctuations' correlation, $C_{k^2}$. For $k_f\sigma \gtrsim 2$, the reflectometry correlation plotted versus $x$-shift, as shown in Fig. 11, rapidly assumes a form that is identical to the form of $C_{k^2}(x)$. This proves to be the case regardless of whether the dominant contribution
is coming from near the cut-off or not. This transformation occurs as a result of the rapid reduction of the total integrated area under the curve $C_{4+}(x)$ (relative to the case when $k_f = 0$). This area, for fixed $\sigma$ is proportional to $\exp(-k_f^2 \sigma^2/2)$. The result is that there is negligible contribution to $\tilde{\phi}$ from the average part of $R(\psi^2/W)$, and the dominant contribution is the beating of the oscillatory part with the fluctuation wave.

A way of understanding this behaviour is to think of the process in scattering terms. One obtains contributions either from forward scattering or from backward scattering (as mentioned by Zou, 1991). The selection rules for these processes are $k_n = 0$ and $k_n = 2k$ respectively, where $k_n$ is the fluctuation-spectrum wave-number. The relative magnitude of the fluctuation spectrum at $k_n = 0$ for the correlation function of Eq.(44) is $\exp(-k_f^2 \sigma^2/2)$. Thus the magnitude of $k_f \sigma$ determines whether the forward scattering, $k_n = 0$, is significant or not. When it is not, there is close agreement between the phase correlation coefficient and the fluctuation correlation coefficient. It is clear that this backward scattering component cannot be described using the smooth WKBJ estimate but requires a full-wave analysis such as this.

The scattering viewpoint also helps to understand the observations of the previous section, regarding monotonic correlation coefficients. The forward-scattering process selects the component $k_n \approx 0$ from the fluctuation spectrum. This component has a long correlation length, and therefore the phase correlation displays this long length even though the total $k_n$-spectrum has a much shorter correlation length. The case $\sigma \approx 0$, which gives a wide $k_n$-spectrum, leads to substantial additional contribution from backscattering, at $k_n = 2k$. It is this contribution that causes the correlation coefficient to have a narrow feature at low $\sigma$. When a substantial imaginary part of $k^2$ is included, it enhances the relative importance of the backscattered component, and thus reduces the phase correlation for small $\sigma$.

For coherent waves, calculations with substantial imaginary part of $k^2$ give virtually the same results for the correlation coefficients as for real $k^2$. Presumably this is because
the signal is already dominated by backscattering.

Thus it seems that if one observes a highly oscillatory correlation function of $\tilde{\phi}$, corresponding to a coherent wave, then there is expected to be close correspondence between the form of the reflectometry correlation coefficient and that of the fluctuations. Moreover, one can identify the localization of the wave being observed as the position where the backscattering selection rule, $k_f \approx 2k$, is satisfied.

7. Summary

The perturbation to the phase of a reflectometer due to perturbations of $k^2$ is equal to an integral (Eq(22)) of the $k^2$ perturbation times a weighting function that is proportional to the square of the solution to the unperturbed problem. From this result, the group delay can immediately be deduced.

This full-wave weighting function has an average in the oscillatory region that is equal to the WKBJ result, but avoids the singularity at the cut-off point. The width of the last lobe of the weighting function is approximately $1.63/|k^{2'}|^\frac{1}{2}$.

Correlation reflectometry must make use of quadrature information, otherwise the variation of fluctuation-sensitivity proportional to the sine of the unperturbed phase will render the signal correlations meaningless. Even with a proper phase signal, turbulent fluctuations with monotonic correlation coefficients give reflectometry correlation coefficients that only distantly reflect the correlation coefficients of the fluctuations. In particular, the correlation observed should be substantial even for shifts much greater than the correlation length, unless the wave attenuation effects are important. In this latter case, low correlation can occur, but mostly because of short-wavelength contributions well away from the cut-off position.

Coherent waves, represented by non-monotonic correlation coefficients, contribute most from a position where their wavelength is equal to half the wavelength of the unperturbed solution. The reflectometry correlation coefficient in this case can closely resemble
the correlation coefficient of the fluctuations.

The distinction between these two cases is fundamentally whether forward or backward scattering contributions dominate. The coherent wave case corresponds to negligible forward scattering and can not be represented using the WKBJ analysis.

The localization of the reflectometry contribution, whether of coherent or incoherent fluctuations, is never to a region narrower than about one (local) wavelength of the unperturbed wave function. And since the vacuum-wavelength is the smallest wavelength the localization is a fortiori never narrower than the vacuum wavelength. It is therefore incorrect, on the basis of this analysis, to interpret the observation of reflectometry correlation lengths less than the vacuum wavelength as evidence for localization of the measurement sensitivity.

Finally, one should emphasize that these conclusions are based on a one-dimensional analysis. Inherently multi-dimensional effects might predominate in actual experiments. Therefore great caution should be exercised in inferring the nature of the fluctuations from reflectometry measurements.

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References


25


Figure Captions

Fig. 1  An example profile of (a) the squared wavenumber, $k^2$, (b) the resulting solution, $\psi_1$, and (c) the weighting function, $\psi_T^2/W$ compared with the corresponding WKBJ approximation $1/2k$.

Fig. 2  The integral of the weighting function (solid line) compared with the corresponding quantity from the WKBJ approximation (dashed line). The spatial coordinate is in scaled units. Physical position is given by $x/|k^2|^{1/2}$.

Fig. 3  The unperturbed solution, $\psi_1$, and the weighting function, $R(\psi_T^2/W)$, for three values of the imaginary part of $k^2$: (a) 0.1, (b) 0.2, (c) 0.3.

Fig. 4  The correlation function for plasma edge a distance $x_e$ from the cut-off position of the higher frequency wave. Curves are labelled with the value of the shift of the lower-frequency cut-off. Three correlation lengths are shown: (a) $\sigma = 0$ (b) $\sigma = 0.3$ (c) $\sigma = 1$. The spatial coordinate is in scaled units. Physical position is given by $x_e/|k^2|^{1/2}$.

Fig. 5  The same as for Fig. 4 except showing the correlation coefficient, $\rho_\phi$.

Fig. 6  The correlation coefficient, $\rho_\phi$, versus x-shift of the lower frequency cut-off position (solid line). The labels indicate the width, $\sigma$, of the corresponding fluctuation correlation coefficient, and the dashed lines plot its form.

Fig. 7  The same as Fig. 4 except that the imaginary part of $k^2$ is 0.3.

Fig. 8  The same as Fig. 5 except that the imaginary part of $k^2$ is 0.3.

Fig. 9  The same as Fig. 6 except that the imaginary part of $k^2$ is 0.3.

Fig. 10 Correlation function versus edge plasma position for coherent waves. Curves are labelled with the x-shift of the reflectometer channels. Total width, $\sigma$, is 3.0. Wave numbers, $k_f$, are (a) 0, (b) 1, (c) 5.

Fig. 11 Reflectometry correlation coefficient versus x-shift for coherent waves. The width, $\sigma$ is 3.0, and curves are labelled with the value of the fluctuation wave-number, $k_f$. The dashed lines are the correlation coefficients of the fluctuations.
Fig. 1
\[ X_p \left( M/\int_0^1 \Phi \right) \]
(\mathbf{\tau}^\lambda)\mathbf{X}

(\mathbf{M}/\mathbf{\zeta}\mathbf{\tau}^\lambda)\mathbf{X}
Correlation Coefficient

Fig. 5(b)
Fig. 6

Correlation Coefficient

x-shift

0.0 0.2 0.4 0.6 0.8 1.0

0.0 0.4 0.8 1.2 1.6 2.0
Correlation Function

Fig. 10(a)
Correlation Function