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Plasmoid instability in the semi-collisional regime

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We investigate analytically and numerically the semi-collisional regime of the plasmoid instability, defined by the inequality $\delta_{SP} \gg \rho_s \gg \delta_{in}$, where $\delta_{SP}$ is the width of a Sweet-Parker current sheet, $\rho_s$ is the ion sound Larmor radius, and $\delta_{in}$ is width of boundary layer that arises in the plasmoid instability analysis. Theoretically, this regime is predicted to exist if the Lundquist number $S$ and the length of the current sheet $L$ are such that $(L/\rho_s)^{14/9} < S < (L/\rho_s)^2$ (for a sinusoidal-like magnetic configuration; for a Harris-type sheet the lower bound is replaced with $(L/\rho_s)^{8/5}$). These bounds are validated numerically by means of simulations using a reduced gyrokinetic model (Zocco & Schekochihin, \textit{Physics of Plasmas} 18, 2011) conducted with the code \textit{Viriato}. Importantly, this regime is conjectured to allow for plasmoid formation at relatively low, experimentally accessible, values of the Lundquist number. Our simulations obtain plasmoid instability at values of $S$ as low as $\sim 250$. The simulations do not prescribe a Sweet-Parker sheet; rather, one is formed self-consistently during the nonlinear evolution of the initial tearing mode configuration. This proves that this regime of the plasmoid instability is realizable, at least at the relatively low values of the Lundquist number that are accessible to current dedicated experiments.

1. Introduction

Magnetic reconnection is a fundamental plasma physics phenomenon, relevant to laboratory, space, and astrophysical systems (Biskamp 2000; Zweibel & Yamada 2008; Uzdensky 2011). It involves a rapid topological rearrangement of the magnetic field, leading to efficient magnetic energy conversion and dissipation. Solar flares (Shibata & Magara 2011) and sawtooth crashes in tokamaks (Hastie 1997) are two popular examples of processes where reconnection plays a key role; others include substorms in the Earth’s magnetosphere (Dungev 1961; Burch \textit{et al.} 2016), particle acceleration in jets and pulsar winds (Kagan \textit{et al.} 2015; Werner \textit{et al.} 2016), magnetized turbulence (e.g. Matthaeus & Lamkin (1986); Servidio \textit{et al.} 2009; Loureiro & Boldyrev 2017; Mallet \textit{et al.} 2017; Cerri & Califano 2017), etc.

In magnetohydrodynamic (MHD) plasmas, reconnection sites (current sheets) tend to be unstable to the formation of multiple small islands (or plasmoids) provided that the Lundquist number (defined as $S = LV_A/\eta$, where $L$ is the system size, $V_A$ the Alfvén speed, and $\eta$ the magnetic diffusivity) is sufficiently large (typically, $S \gtrsim 10^4$). This is known as the plasmoid instability; the current sheets mediated by the plasmoids have an aspect ratio that is much smaller than that of the global sheet, thus triggering fast reconnection (Loureiro \textit{et al.} 2007; Lapenta 2008; Bhattacharjee \textit{et al.} 2009).

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In weakly collisional plasmas, where the frozen-flux constraint is broken by kinetic effects instead of collisions, plasmoids are also abundantly observed (e.g. Drake et al. (2006); Ji & Daughton (2011); Daughton & Roytershteyn (2012)), suggesting that plasmoid generation and dynamics are robust and fundamental features of reconnecting systems, regardless of the collisionality of the ambient plasma.

Because of its perceived importance — from determining the reconnection rate in MHD plasmas to its possible role on the reconnection onset (Pucci & Velli 2014; Uzdensky & Loureiro 2016; Comisso et al. 2016; Tolman et al. 2018) and in the energy partition (Loureiro et al. 2012; Numata & Loureiro 2015) and particle acceleration (Drake et al. 2006; Oka et al. 2010; Cerutti et al. 2012; Sironi & Spitkovsky 2014; Guo et al. 2015; Zhou et al. 2015; Sharma et al. 2017) — the plasmoid instability has been the subject of a multitude of theoretical and numerical studies (see Loureiro & Uzdensky (2016) for a brief review). There are also abundant reports of plasmoid observation in solar flares (Nishizuka et al. 2010; Milligan et al. 2010; Liu et al. 2013), coronal jets (Zhang & Ji 2014) and in the Earth’s magnetotail (Moldwin & Hughes 1992; Zong et al. 2004) and other planets (Jackman et al. 2011; Zhang et al. 2012; DiBraccio et al. 2015). This, however, contrasts starkly with plasmoid detection and investigation in laboratory experiments, which has so far been relatively limited, with only a handful of studies reporting plasmoid observation (Fox et al. 2011; Dorfman et al. 2013; Olson et al. 2016; Jara-Almonte et al. 2016; Hare et al. 2017). In all cases, these observations have occurred in non-MHD regions of parameter space (dedicated reconnection experiments have not been able to reach $S > 10^4$, though future ones might (Ji et al. 2015; Ekedal et al. 2014)), and lack a solid theoretical footing.

In a recent paper, Loureiro & Uzdensky (2016) have identified a plasma collisionality regime where the requirement for triggering the plasmoid instability is significantly eased with respect to its pure MHD counterpart. In essence, this regime relies on collisionality being high enough that an MHD current sheet may form in the first place (i.e., the current sheet thickness exceeds any kinetic scale); but small enough that when such a sheet is analyzed for its stability to plasmoid formation, two fluid effects can no longer be neglected. Interestingly, some of the above mentioned experimental reports of plasmoid detection (Dorfman et al. 2013; Jara-Almonte et al. 2016; Hare et al. 2017) seem to sit in, or very close to, this region of parameter space (Hare 2017), and it is conceivable that they provide experimental evidence of the existence of this novel regime.

The aim of this paper is to report a set of numerical experiments designed to confirm the existence of the semi-collisional plasmoid instability, with particular focus on experimentally accessible values of the Lundquist number, and precisely map out the regions of parameter space inhabited by it.

2. The semi-collisional plasmoid instability

The linear theory of the plasmoid instability in MHD plasmas (Loureiro et al. 2007; Bhattacharjee et al. 2009; Loureiro et al. 2013) assumes the existence of a Sweet-Parker (SP) current sheet (Sweet 1958; Parker 1957), which is taken as the background equilibrium whose stability is analyzed. In standard tearing mode fashion (Furth et al. 1963), the calculation divides the spatial domain into an outer region, where resistivity effects
can be ignored, and an inner region — a boundary layer of thickness $\delta_{in}$ — where resistivity matters.

Let us revisit this question adding minimal kinetic effects: we wish to consider the case where the ion sound Larmor radius, $\rho_s$, though smaller than the thickness of the SP current sheet, $\delta_{SP}$, is however larger than the boundary layer of the MHD linear plasmoid instability: $\delta_{SP} \gg \rho_s \gg \delta_{in}$. This obviously implies that MHD is no longer a sufficient description: ion kinetic effects need to be taken into account.

The expressions for the growth rate of the plasmoid instability in this regime can be obtained from the appropriate tearing mode theory [Drake & Lee 1977; Pegoraro & Schep 1986; Zocco & Schekochihin 2011], following the usual procedure of replacing the equilibrium scale length with $\delta_{SP} \sim L S^{-1/2}$, where $L$ is the length of the current sheet [Tajima & Shibata 2002; Bhattacharjee et al. 2009].

For small values of the tearing mode instability parameter $\Delta'$, i.e., $\Delta' \delta_{in} \ll 1$, we find

$$\gamma L/V_A \sim (kL)^{2/3}(\Delta' \rho_s)^{2/3}, \quad (2.1)$$

$$\delta_{in}/L \sim (\Delta' \rho_s)^{1/6}(kL)^{-1/3}S^{-1/2}, \quad (2.2)$$

where $\gamma$ is the growth rate of a mode with wavenumber $k$. This expression can be simplified if $\Delta'$ is not too small, such that it can be approximated as $\Delta' \delta_{SP} \sim 1/(k \delta_{SP})$, as pertains to the usual Harris-like magnetic configuration [Harris 1962]. In that case, we obtain

$$\gamma L/V_A \sim (\rho_s/L)^{2/3}S^{2/3}, \quad (2.3)$$

and the validity condition $\Delta' \delta_{in} \ll 1$ becomes $kL \gg (\rho_s/L)^{1/9}S^{4/9}$. Note that this expression is independent of $k$ to lowest order.

In the opposite limit of large $\Delta'$, i.e., $\Delta' \delta_{in} \gg 1$, we instead have

$$\gamma L/V_A \sim (\rho_s/L)^{4/7}S^{2/7}(kL)^{6/7}, \quad (2.4)$$

$$\delta_{in}/L \sim (\rho_s/L)^{1/7}(kL)^{-2/7}S^{-3/7}. \quad (2.5)$$

The fastest growing mode is yielded by the intersection of these two branches [Loureiro & Uzdensky 2016].

$$\gamma_{max} L/V_A \sim (\rho_s/L)^{2/3}S^{2/3}, \quad (2.6)$$

$$k_{max} L \sim (\rho_s/L)^{1/9}S^{4/9}, \quad (2.7)$$

$$\delta_{in}/L \sim (\rho_s/L)^{1/9}S^{-5/9}. \quad (2.8)$$

The validity of these expressions rests on two conditions: $\delta_{SP} \gg \rho_s$, and $\rho_s \gg \delta_{in}$. Using Eq. (2.8), these therefore imply that the semi-collisional plasmoid instability inhabits the region of parameter space defined by

$$(L/\rho_s)^2 \gg S \gg (L/\rho_s)^{8/5}. \quad (2.9)$$

An alternative current sheet profile worth considering – and the one we will make use

† An additional requirement is that the electron skin depth, $d_e = c/\omega_{pe}$, with $\omega_{pe}$ the electron plasma frequency, is negligible, i.e., $\delta_{in} \gg d_e$. A further generalization of the theory to include electron inertia effects is possible — see [Loureiro & Uzdensky 2016].

‡ Note that the scaling for $\delta_{in}$, Eq. (2.8), has been corrected from [Loureiro & Uzdensky 2016].

¶ In addition, the existence of the plasmoid instability (irrespective of the collisionality regime) requires that $\gamma L/V_A \gg 1$ and $kL/V_A \gg 1$; both of these conditions yield requirements on $S$ and $L/\rho_s$ that are less demanding than the rightmost inequality in (2.9), so $(L/\rho_s)^{8/5}$ should be the correct lower bound.
Figure 1. Dispersion relation for the semi-collisional plasmoid instability for two different combinations of the relevant parameters: \((S, L/\rho_s) = (1000, 35)\) (red lines) and \((S, L/\rho_s) = (10^6, 4000)\) (blue lines). Solid lines represent the case of \(\Delta'\delta_{in} \ll 1\) (Eq. (2.11)), whereas dashed lines are for the case of \(\Delta'\delta_{in} \gg 1\) (Eq. (2.4)). Respectively, these lines are only valid to the right, or to the left, of their intersection. Their point of intersection provides an estimate of the most unstable wavenumber and corresponding growth rate.

of in this paper – is that of a sinusoidal-like magnetic field, for which \(\Delta'\delta_{SP} \sim 1/(k\delta_{SP})^2\). For small \(\Delta'\), we obtain,

\[
\gamma L/V_A \sim (\rho_s/L)^{2/3}(kL)^{-2/3}S, \quad \delta_{in} \sim (\rho_s/L)^{1/6}(kL)^{-2/3}S^{-1/4},
\]

valid if \(kL \gg (\rho_s/L)^{1/16}S^{15/32}\). For large \(\Delta'\), the scaling for \(\gamma L/V_A\) is same as in Eq. (2.4), as there is no explicit dependence on \(\Delta'\). The fastest growing mode is therefore characterized by

\[
\gamma_{max} L/V_A \sim (\rho_s/L)^{5/8}S^{11/16}, \quad k_{max} L \sim (\rho_s/L)^{1/16}S^{15/32}, \quad \delta_{in}/L \sim (\rho_s/L)^{1/8}S^{-9/16}.
\]

Fig. 1 illustrates both limits of the dispersion relation, and their intersection, for two different combinations of the two relevant parameters, \(S\) and \(L/\rho_s\). In appendix A we recover these scalings via direct numerical simulation, confirming both their validity and the ability of the code \textit{Viriato} (Loureiro et al. 2016), which we will employ in this paper (see Sections 3 and 4), to recover them.

In this case of sinusoidal-like current sheet profile, Eq. (2.9) is replaced by:

\[
(L/\rho_s)^2 \gg S \gg (L/\rho_s)^{14/9}.
\]

The lower bound here has only a slightly smaller exponent than (and in practice difficult to discern from) the Harris-like case of Eq. (2.9).

Eqs. (2.9) and (2.15) lead to the interesting suggestion that this particular version of the plasmoid instability can be obtained at relatively low values of \(S\), provided that the system (the current sheet length \(L\), to be precise) is not too large compared with \(\rho_s\). In other words, the lower bound of \(S \sim 10^4\) that pertains to the MHD version of the plasmoid instability (Biskamp 1986; Loureiro et al. 2005; Samtaney et al. 2009; Baty 2014) is replaced by a function in the semi-collisional regime, \((L/\rho_s)^{14/9}\), or \((L/\rho_s)^{8/5}\), as appro-
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From both the experimental and the numerical points of view, this is a significant advantage. In particular, this regime should be available to reconnection experiments such as MRX, TREX, FLARE and Magpie (Ji & Daughton 2011; Loureiro & Uzdensky 2016; Hare et al. 2017a, b, 2018). Indeed, as we mention above, it is tempting to attribute recent reports of experimental plasmoid observation (Dorfman et al. 2013; Jara-Almonte et al. 2016; Hare et al. 2017a, b, 2018) to this version of the plasmoid instability — or to its \( \beta \approx 1 \) analogue (Baalrud et al. 2011) (\( \beta \) is the ratio of the plasma pressure to the magnetic pressure).

Despite these speculations and conjectures, the existence of this regime of the plasmoid instability has not been confirmed via direct numerical simulations, and indeed there are a couple of issues that may raise suspicion. In particular, we note that all of the scalings above are predicated on there being an asymptotic separation between the scales involved, namely \( \delta_{SP} \gg \rho_s \gg \delta_{in} \). As the Lundquist number is made smaller, this scale separation is inevitably lost, and thus the claim that the semi-collisional plasmoid instability may be obtainable at the relatively low values of \( S \) and \( L/\rho_s \) that are within the reach of existing experiments needs careful numerical validation.

An additional concern of significant relevance is that of whether the Sweet-Parker sheet that is assumed as the background for the instability derived here is realizable. As has been pointed out (Loureiro et al. 2007; Pucci & Velli 2014; Uzdensky & Loureiro 2016), the existence of a super-Alfvénic instability whose growth rate diverges as \( S \to \infty \) indicates that the equilibrium that it arises from, may never form in the first place. We think this claim is pertinent at high values of the Lundquist number. However, in the opposite limit of relatively low \( S \) with which we are mostly concerned here, where the instability is only mildly super-Alfvénic, a Sweet-Parker sheet may still form and beget the instability.

This paper aims to answer these questions by means of direct numerical simulations of a reconnecting system where the current sheet is not prescribed, but rather allowed to form self-consistently.

3. The model

The weak collisionality of a large number of reconnecting environments demands the use of a kinetic description. In the most general case, one is forced to adopt a first-principles formalism (such as particle-in-cell, or the 6D Vlasov (or Boltzmann) equation), with its inherent analytical and numerical complexity. In many situations, however, a strong component of the magnetic field is present that is perpendicular to the

† Note that the condition for semi-collisional regime should not be confused with a similar looking criterion specified in a scenario of hierarchy of plasmosids and interplasmoid current sheets, namely, \( \delta_{SP}(L) > \rho_s > \delta_c \) (see Uzdensky et al. 2010). Here, \( \delta_{SP}(L) \) refers to the width of the primary global SP current sheet and \( \delta_c \) is the width of the smallest interplasmoid SP current sheet which is marginally unstable to the plasmoid instability. This criterion guarantees a transition of the plasmoid hierarchy from MHD to kinetic scales.

The scalings Eqs. (2.8) and (2.14) allow us to make this criterion more precise. Consider the plasmoid cascade. For an interplasmoid current sheet with the width \( \delta^{(N)} \) (\( \delta^{(2)} \), \( \delta^{(3)} \) are the secondary and tertiary SP current sheets, and so on), if the corresponding inner layer \( \delta^{(N)}_{in} \) is larger than \( \rho_s \), then the arising plasmosids are still in MHD regime. However, if at any point in the plasmoid cascade one obtains \( \rho_s > \delta^{(N)}_{in} \), then the system transitions into the semi-collisional regime. Using the relationship \( \delta_c \sim L_c S_c^{-1/2} \), where \( S_c \sim 10^4 \) is the critical value of the Lundquist number to obtain the plasmoid instability in MHD, it is easy to conclude that a transition to the semi-collisional regime must occur at some point in the plasmoid hierarchical cascade if \( S > S_c^{3/8}(L/\rho_s) \), for a Harris-type sheet, or \( S > S_c^{5/14}(L/\rho_s) \), for a sinusoidal-type sheet.
reconnection plane. This guide-field offers an opportunity for significant simplification: the 5D gyrokinetic formalism \cite{Frieman1982,Howes2006}.

Further simplification is possible if, in addition, one considers plasmas such that the electron beta is sufficiently low to be comparable to the electron-to-ion mass ratio \((m_e/m_i)\); a case in point is the solar corona, as mentioned earlier in section \[2\]. Leveraging on these assumptions (strong guide-field and low beta), a reduced-gyrokinetic formalism was recently derived (dubbed ‘Kinetic Reduced Electron Heating Model’, or KREHM) \cite{Zocco2011}. One of its appealing features is that the phase-space is reduced further to 4D (position vector and velocity in the guide-field direction only), thus rendering computations, and even analytic theory, significantly more manageable than fully kinetic approaches.

In this work, we use the KREHM equations to investigate the plasmoid instability in the semi-collisional regime. We will restrict our numerical investigations to the two spatial dimensions comprising the reconnection plane, \((x, y)\). With this restriction, the KREHM equations become:

\[
\frac{1}{n_{0e}} \frac{d\delta n_e}{dt} = \frac{1}{B_0} \left\{ A_{||}, \frac{e}{cm_e} d_e^2 \nabla^2_{\perp} A_{||} \right\}, \tag{3.1}
\]

\[
\frac{d}{dt} (A_{||} - d_e^2 \nabla^2_{\perp} A_{||}) = \frac{cT_{0e}}{eB_0} \left\{ A_{||}, \left( \frac{\delta n_e}{n_{0e}} + \frac{\delta T_{||e}}{T_{0e}} \right) \right\}, \tag{3.2}
\]

\[
\frac{dg_e}{dt} - \frac{v_{||}}{B_0} \left\{ A_{||}, \left( g_e - \frac{\delta T_{||e}}{T_{0e}} F_{0e} \right) \right\} = C[g_e] - \left( 1 - \frac{2v_{||}^2}{v_{the}^2} \right) \frac{F_{0e}}{B_0} \left\{ A_{||}, \frac{e}{cm_e} d_e^2 \nabla^2_{\perp} A_{||} \right\}, \tag{3.3}
\]

where

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{c}{B_0} \{ \phi, \ldots \}. \tag{3.4}
\]

Here, \(\phi\) is electrostatic potential and \(\{\ldots\}\) denotes the Poisson bracket, defined for any two fields \(P, Q\) as \(\{P, Q\} = \delta_x P \delta_y Q - \delta_y P \delta_x Q\). The background magnetic guide field is \(B_0\); and \(n_{0e}, T_{0e}\) are the background electron density and temperature, respectively. Other symbols introduced above are the electron skin depth, \(d_e = c/\omega_{pe}\), with \(\omega_{pe} = \sqrt{4\pi n_{0e}e^2/m_e}\) the electron plasma frequency; and the electron thermal speed, \(v_{the} = \sqrt{2T_{0e}/m_e}\), with \(e\) the electron charge, and \(m_e\) its mass.

The perturbed electron density, \(\delta n_e\), and the electrostatic potential are related via the gyrokinetic Poisson law,

\[
\frac{\delta n_e}{n_{0e}} = \frac{1}{\tau} (\hat{T}_0 - 1) e\phi / T_{0e}. \tag{3.5}
\]

where, \(\tau = T_{0i}/T_{0e}\) is the ion to electron background temperature ratio, and \(\hat{T}_0\) is a real space operator whose Fourier transform is \(\hat{T}_0(\alpha) = I_0(\alpha)e^{-\alpha}\), with \(I_0\) the modified Bessel function of zeroth order, and \(\alpha = k^2 \rho_i^2/2\), with \(\rho_i = v_{thi}/\Omega_i\) the ion Larmor radius, \(v_{thi} = \sqrt{2T_{0i}/m_i}\) the ion thermal velocity, and \(\Omega_i = |e|B_0/m_i c\) the ion gyro frequency.

Eqs. (3.1) and (3.2) evolve the zeroth (the electron density perturbation, \(\delta n_e\)) and first (the parallel electron flow, \(\delta n_e/u_{||e}\)) moments of the perturbed electron distribution function, given by

\[
\delta f_e = \delta n_e + \frac{2v_{||}u_{||e}}{v_{the}^2} F_{0e}, \tag{3.6}
\]

where \(F_{0e} = n_{0e}/(\pi/v_{the}^2)^{3/2}\exp[-(v_{||}^2 + v_{\perp}^2)/v_{the}^2]\) is the Maxwellian equilibrium, and
can self-consistently obtain an SP current sheet whose stability to plasmoid formation lead to the dynamic formation of a SP current sheet that meets the conditions of the can then be studied. Specifically, the input parameters can be specified in a way as to be written in terms of \( g \) of the Hermite polynomials, where by construction \( g \) is to demand that at some size \( L \) of the reconnecting field, \( B_\parallel \) can be solved in terms of its expansion in Hermite polynomials: \[ g_e(x, y, z, v_\parallel, t) = \sum_{m=2}^{\infty} \frac{1}{2^{m}m!} H_m(v_\parallel/v_{the})g_m(x, y, z, t)F_0e(v_\parallel), \] (3.7)

where \( H_m \) denotes the Hermite polynomial of order \( m \) and \( g_m \) is its coefficient. Inserting this expansion into Eq. (3.3) yields a coupled set of fluid-like equations for the coefficients of the Hermite polynomials, where by construction \( g_0 = g_1 = 0 \), and for \( m \geq 2 \) we have

\[
\frac{dg_m}{dt} - \frac{v_{the}}{B_0} \left( \sqrt{\frac{m + 1}{2}} \left\{ A_\parallel, g_{m+1} \right\} + \sqrt{\frac{m}{2}} \left\{ A_\parallel, g_{m-1} \right\} \right) = \frac{\sqrt{2} \delta_{m,2}}{B_0} \left\{ A_\parallel, \frac{e}{cm_e} d_e^2 \nabla_\perp^2 A_\parallel \right\} - \nu_{ei}(mg_m - 2\delta_{m,2}g_2),
\] (3.8)

where \( \nu_{ei} \) is the electron-ion collision frequency, \( \nu_{ei} = \eta/d_e^2 \). The kinetic equations solved by means of Hermite expansion requires a closure. A way to close the set of equations is to demand that at some \( m = M \), the collision term becomes significant such that \( g_{M+1}/g_M \ll 1 \). This constraint will truncate the kinetic equations at \( g_M \) as \( g_{M+1} \) can be written in terms of \( g_M \) (Zocco & Schekochihin 2011; Zocco et al. 2015; Loureiro et al. 2016; White & Hazeltine 2017). This type of closure also recovers the semi-collisional limit exactly.

4. Numerical setup

Eqs. (3.1), (3.2), (3.3) and (3.8) are solved numerically on a two-dimensional grid of size \( L_x \times L_y \), using the pseudo-spectral code Viriato (Loureiro et al. 2016). Periodic boundary conditions are employed in both directions. The numerical configuration is akin to that employed in Loureiro et al. (2005) and, as we will show, is such that one can self-consistently obtain an SP current sheet whose stability to plasmoid formation can then be studied. Specifically, the input parameters can be specified in a way as to lead to the dynamic formation of a SP current sheet that meets the conditions of the semi-collisional regime that we have discussed earlier.

The initial equilibrium is \( A_\parallel(x,y,t = 0) = A_\parallel(0) / \cosh^2(x) \), where \( A_\parallel(0) = 3\sqrt{3}/4 \), such that the maximum of the reconnecting field, \( B_y = dA_\parallel/dx \), is \( B_y_{,\text{max}} = 1 \). This equilibrium is destabilized with a small amplitude (linear) perturbation which seeds the fastest growing tearing mode; in all simulations, this is the longest wavelength mode that fits in the simulation box. Once in the nonlinear stage, the tearing mode undergoes X-point collapse (Waelbroeck 1989; Loureiro et al. 2003), and a current sheet forms which
is consistent with the SP scaling (as we shall confirm). The plasmoid instability is then triggered, or not, depending on the values of $S$ and $\rho_s$ specified in the simulation.

The length of the SP current sheet can be varied by changing the instability parameter, $\Delta'(k)$, pertaining to the initial tearing mode (Loureiro et al. 2005). In practice, this is achieved by changing the length of the box in the outflow direction, $L_y$.

The resolution of a simulation in the inflow ($x$) direction is set by the size of the inner boundary layer that is expected to arise due to the plasmoid instability, estimated using Eq. (2.8). The resolution demands in the outflow ($y$) direction are less stringent, and are determined on an ad hoc basis.

An additional constraint on our runs is that the electron inertia play no role. This is insured by setting it to be smaller than the resolution for any given simulation. Therefore, in all runs, the frozen-flux constraint is broken by resistivity. Note that no hyper-resistivity (or hyper-viscosity) is used in these runs, ensuring that the actual Lundquist number in the simulations is determined by the resistivity that we specify. The simulations also employ a finite viscosity, set equal to the magnetic diffusivity.

A final choice has to do with the number of Hermite polynomials to keep in the simulations. For all runs reported here, the highest order polynomial is $M = 4$ — this ought to be sufficient given the relatively high collisionality of our simulations (and indeed we find that in all our runs resistivity is the dominant energy dissipation channel). To check convergence, we performed one test using instead $M = 10$, and observed that in the spectrum of $|g_m|^2/2$, the energy at $m > 3–4$, is lower than that at $m = 2$ by orders of magnitude, indicating that the power transferred to the Hermite polynomials of higher order is not significant. In all runs, the convergence of the Hermite representation is accelerated by the use of a hyper-collision operator (see Loureiro et al. (2016) for details).

5. Results

As previously stated, our main aim is to numerically ascertain the existence of the semi-collisional plasmoid instability and validate the bounds of the parameter space defined by $S$ and $L/\rho_s$, where this instability is expected to be active. We specifically wish to focus on the instability’s existence at modest, experimentally accessible, values of the Lundquist number. To this effect, we perform a series of runs as listed in Table 1. Amongst other parameters, the table lists the length of the current sheet, $L$, that is dynamically obtained during the nonlinear evolution of the primary tearing mode (which results from the collapse of the $X$-point, as previously described). This length is measured using a full width at half maximum estimate (as is the current sheet thickness, $\delta$), and it is this length that is used to estimate the Lundquist number that is also listed in Table 1 (the magnitude of the upstream magnetic field remains unchanged by the $X$-point collapse).

The first step in the description of our results is the characterization of the current sheet that is dynamically obtained from the $X$-point collapse of the primary tearing mode. The theory of the semi-collisional plasmoid instability laid out in Section 2 assumes a SP sheet as the background equilibrium; and so it is important to determine whether indeed that is the case in our simulations. In Fig. 2, we plot $\delta/L$ as a function of the Lundquist number $S$, obtained from all the runs. We find, as shown in Fig. 2 by the blue stars (Runs A to H) and red diamonds (Runs I, J, K), that the current sheets in these runs follow the SP scaling. The system in these runs is initially purely in the MHD regime as the inner boundary layer thickness of the primary tearing mode is larger than the kinetic scales. And thus, upon $X$-point collapse of the MHD tearing mode, the current sheets that form are expected to follow the SP scaling (Loureiro et al. 2005), which indeed bears out.
Table 1. Summary of all the runs discussed in the paper. The table lists the values of the sound Larmor radius $\rho_s$, length of the current sheet $L$, their ratio $L/\rho_s$, Lundquist number $S$, the inner boundary layer width $\delta_{in}$ (using Eq. (2.14)), the number of grid points employed $N_x \times N_y$, the length of domain in the $y$-direction, $L_y$, resistivity $\eta$, and answers whether both constraints of the semi-collisional regime are satisfied or not for a given run.

<table>
<thead>
<tr>
<th>Run</th>
<th>$\rho_s$</th>
<th>$L$</th>
<th>$L/\rho_s$</th>
<th>$S$</th>
<th>$\delta_{in}$</th>
<th>$N_x \times N_y$</th>
<th>$L_y/2\pi$</th>
<th>$\eta$</th>
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Figure 2. The ratio of current sheet width $\delta$ to its length $L$ against the Lundquist number $S$ is shown for Runs A through H (blue stars), I, J and K (red diamonds), and L, M and N (black inverted triangles). Refer to Table 1 for details of these runs. The dotted line indicates an $S^{-1/2}$ slope, expected for Sweet-Parker current sheets.

In the case of the three black inverted triangles shown in the Fig. 2 corresponding to Runs L, M and N, the ion sound Larmor radius $\rho_s$ is set to be larger than the SP current sheet width ($\delta_{SP}$) (as expected in the MHD case). Thus, when X-point collapse
Figure 3. Reconnection phase diagram showing the semi-collisional regime marked out by the upper bound ($\delta_{SP} > \rho_s$; solid line) and lower bound ($\rho_s > \delta_{in}$; dashed line) as prescribed by Eq. (2.15). Blue stars denote simulations which show the plasmoid instability; red diamonds and black inverted triangles correspond to simulations which do not. The parameters (and other details) of each run can be looked up in Table 1.

happens, these runs are affected by ion kinetic physics; unsurprisingly, the current sheets here do not follow the SP scaling.

Next, we show whether or not the plasmoid instability is observed for a given run on the $S$-$\rho_s$ parameter space in Fig. 3. The two groups of runs represented by blue stars and red diamonds, which form SP current sheets, have $\rho_s < \delta_{SP}$ (the symbols here match those in Fig. 2). Furthermore, in the case of the blue starred runs, upon the formation of the current sheet, the system becomes sensitive to the presence of $\rho_s$, as this kinetic scale is now larger than the semi-collisional inner boundary layer $\delta_{in}$ (see Eq. (2.8)). As a result, the plasmoid instability can arise, and indeed it does, as seen in Runs A to H (blue stars). On the other hand, in the case of Runs I, J, K (red diamonds), $\rho_s$ is not only smaller that $\delta_{SP}$, but is also smaller that $\delta_{in}$ and thus no plasmoids arise in these runs. The condition of $\rho_s < \delta_{in}$ implies the absence of the plasmoid instability simply because the system continues to be in MHD regime and $S$ is much below the critical value of $\sim 10^4$ required in MHD.

The group of runs A to H which do result in plasmoid instability are within the two theoretical bounds of the semi-collisional regime marked by solid and dashed black lines (Eq. (2.15)). Runs I, J and K (where $\rho_s < \delta_{in}$) below the lower bound match roughly in $S$ with Runs A, C and E respectively; Runs L, M, N (where $\rho_s > \delta_{SP}$) above the upper bound, were performed to match in $S$ roughly with Runs A, B and E respectively. We find that neither of these two sets of runs yield the plasmoid instability. We conclude, therefore, that the theoretically prescribed bounds demarcating the semi-
Plasmoid instability in the semi-collisional regime

Figure 4. Contour plots of $A_\parallel$ for Run A (top row) with lowest $S = 256$ (and $L/\rho_s = 25$), and Run H (bottom row) with highest $S = 5560$ (and $L/\rho_s = 164$) at three different times: left panel, just before the plasmoid forms; middle panel, in the early stages of plasmoid formation; and right panel, when the plasmoid is in its nonlinear stage.

collisional regime are remarkably robust within the range of Lundquist numbers and Larmor radii that we have explored.

Further insight can be gained by visually analyzing these runs. In particular it is of interest to compare and contrast runs A and H, corresponding, respectively, to the lowest ($S = 256$) and highest ($S = 5560$) values of the Lundquist number at which we have observed the plasmoid instability. Fig. 4 shows contour plots of $A_\parallel$ at three different times. The two leftmost panels (top and bottom) depict the system just before the plasmoid formation. The middle panels show the early stages of the plasmoid development; and the rightmost panels show the plasmoid well into the nonlinear stage. In both cases, note that in its early stage (middle panel) the $y$-extent of the plasmoid (roughly its linear wavelength) is much smaller than the length of the current sheet (to be discussed below). Also, due to the highly symmetric configuration of the magnetic field (and the intrinsic symmetry germane to the pseudo-spectral method that we employ), the plasmoid is stuck to the middle of the sheet. In a less constrained situation, we expect that this plasmoid would be ejected upwards or downwards, and subsequent plasmoids to be seeded until the system approaches saturation.

Another interesting comparison is between Runs E ($S = 1149$, $L/\rho_s = 70.5$) and K ($S = 1092$, $L/\rho_s = 95$). As seen in Fig. 5, Run E shows a similar time evolution to that displayed by the runs in Fig. 4, with the current sheet becoming unstable to plasmoid formation. In Run K, on the other hand, the collapsed current sheet never goes unstable, and the primary tearing mode just proceeds to saturation. This is rather remarkable given how close in the phase-space outlined by $S$ and $L/\rho_s$ these two runs are (see Fig. 5).

It is noteworthy that all of the runs in this regime are at Lundquist numbers much lower than the MHD critical value of order $\sim 10^4$. The lowest $S$ at which a simulation (Run A) obtains the plasmoid instability is 250. As one decreases the value of $S$ even further we find in our simulations that width of the primary island becomes as large as
Figure 5. Contour plots of $A_\parallel$ for Run E (top row), with $S = 1149$ and $L/\rho_s = 70.5$; and for Run K (bottom row), with $S = 1092$ and $L/\rho_s = 95$, at three consecutive times.

Figure 6. Wavenumber measured from the width of the arising plasmoid in runs A-G, plotted against the theoretical prediction Eq. (2.13) the simulation domain before any plasmoid is observed. In a less constrained system it is possible that the plasmoid instability remains active at even lower values of $S$.

An intriguing observation is that only one single plasmoid arises in the runs A-H. These runs range over a decade in Lundquist number and thus the expected number of plasmoids varies by a factor of $\gtrsim 3$ from the run with lowest $S$ to the run with highest $S$, according to Eq. (2.13). That is not what we obtain, suggesting disagreement with the linear predictions. This leads one to wonder why it is that the lower bound, $S > (L/\rho_s)^{14/9}$, is validated so well (Fig. 3), given that its validity relies on the scalings for $k_{\text{max}}$ (and $\gamma_{\text{max}}$) being correct.

To address this issue, we proceed as follows. For each run, instead of counting the
number of plasmoids to estimate the wavenumber, we compute it from the measurement of the y-extent of the plasmoid (full width at half maximum) at the earliest possible stages of its appearance. We find that this measurement of the wavenumber does follow the theoretical scaling rather well, as shown in Fig. 6. We suspect that the explanation for this result lies in the effects of spatial inhomogeneity in the direction along the sheet, as well as flows and reconnected component of the magnetic field. None of these effects is negligible at these low values of the Lundquist number, but they are all neglected in the theoretical derivation. This argument is strengthened further by the results shown in appendix A. The important observation, however, is that the plasmoid that does form has the correct wavenumber as predicted by linear theory.

6. Discussions and conclusions

We have investigated the plasmoid instability in the semi-collisional regime using 2D reduced-kinetic simulations. This regime is analytically predicted to occupy a significant sliver of the reconnection phase diagram defined by a lower bound $S > (L/\rho_s)^{14/9}$ and an upper bound $S < (L/\rho_s)^2$. Our numerical simulations show that these bounds are remarkably robust: runs which fall within these two bounds yield the plasmoid instability, and vice versa. The instability arises at much lower values of the Lundquist numbers than the MHD analog; as low as $S \sim 250$. We are limited in our exploration of even lower values of $S$ by our simulation setup. Thus, we do not rule out the formation of plasmoids for $S < 250$ in less constrained systems; indeed, we speculate that this regime could also potentially explain formation of plasmoids in recent reconnection experiments (Jara-Almonte et al. 2016, Hare et al. 2017a, b, 2018).

The numerical experiments reported here are limited in that the amount of flux to reconnect is finite, and the simulation box is periodic. As such, (statistical) steady state reconnection cannot be attained, thereby preventing us from numerically answering the important question of what the reconnection rate is in the semi-collisional plasmoid regime. However, theoretically we may expect the following. In the phase-space diagram of Fig. 3, assume that the initial system, with a certain $S$, $L$ and $\rho_s$, is in the semi-collisional regime. As the plasmoid instability unfolds, we expect that smaller, inter-plasmoid current sheets will arise. These will necessarily have a smaller length, $L' \sim L/N$, where $N$ is the number of primary plasmoids. Each of these inter-plasmoid current sheets now defines a reconnecting site which can be located in the reconnection phase-diagram. Since $\rho_s$ and $\eta$ are fixed, and assuming that $V_A$ is the same in these daughter sheets, the only parameter that has changed is the length, from $L$ to $L'$. This means a diagonal displacement in the direction of smaller $L/\rho_s$ from the initial point in that diagram; the slope of that diagonal is 1, because both axes are linearly proportional to $L$. If the new position in this diagram remains in the semi-collisional regime, each inter-plasmoid current sheet is still unstable to the semi-collisional plasmoid instability. The process then repeats (i.e., the plasmoid hierarchy unfolds further) until arriving at an inter-plasmoid current sheet which is now short enough to be outside of the semi-collisional bounds. Inevitably, therefore, this lands the system to the left of the $S \sim (L/\rho_s)^2$ bound, i.e., the collisionless regime, where the expected reconnection rate is $\sim 0.1\tau_A^{-1}$. Defining $\lambda_c = (\rho_s/L)c$ as an empirical scale separating the single from the multiple X-line collisionless regimes (numerically observed to be $\sim 50$, see Ji & Daughton (2011)), we conclude that the system finally lands in either the multiple, or single, X-line collisionless regime depending on whether it is initially above or below the diagonal line $S \sim \lambda_c(L/\rho_s)$, (which intersects the line $S \sim (L/\rho_s)^2$ at $(L/\rho_s) = \lambda_c$) in the reconnection phase space diagram.
Finally, let us outline some general arguments for the case when, unlike our simulations, the global Lundquist number of the system is so large that a Sweet-Parker current sheet may not be able to form dynamically. Consider then a forming current sheet, and assume for simplicity that the characteristic time at which it is forming is Alfvénic, \( \tau_A = L/V_A \).

At any given moment of time, we parameterize the forming current sheet aspect ratio as \( \alpha = \frac{a}{L} \), where \( a \) is the current sheet width, and \( \alpha \) is a number such that \( 0 < \alpha < 1/2 \), with \( \alpha = 1/2 \) representing a Sweet-Parker sheet (which the system presumably never gets to).

Using (2.6) (for a Harris-type sheet) we see that the growth rate of the most unstable semi-collisional mode exceeds the current sheet formation rate (Alfvénic) when

\[
\gamma_{max} \tau_A \sim \left( \frac{\rho_s}{L} \right)^{2/3} S^{-1/3+2\alpha} > 1 .
\]

This expression constrains the relationship between \( S \) and \( L/\rho_s \), for any value of \( \alpha \), to attain Alfvénic growth at that value of \( \alpha \). In addition, we must further require that this mode (the most unstable semi-collisional mode) is indeed in the semi-collisional regime, i.e., \( \delta_{in,max}^S < \rho_s \). This leads to,

\[
\left( \frac{\rho_s}{L} \right)^{1/9} S^{-(2\alpha/3)−2/9} < \rho_s/L .
\]

These two expressions intersect when \( \alpha = 1/3 \). That is, if \( \alpha < 1/3 \), the relationship that \( S \) and \( L/\rho_s \) must satisfy to yield Alfvénic growth is given by (6.1); if instead \( \alpha > 1/3 \), then it is sufficient to satisfy (6.2) to attain Alfvénic growth.

A detailed analysis of all different possibilities that are encountered as \( \alpha \) increases is beyond the scope of this paper and will be left to future work. Generally, for \( \alpha < 1/3 \), the condition for the fastest semi-collisional mode to become faster than Alfvénic becomes progressively less demanding on \( S \). Consider the particularly interesting case where \( \alpha \) reaches the value of 1/3. At this value, we must have \( S > (L/\rho_s)^2 \) for the semi-collisional mode to both exist and be super-Alfvénic. Remarkably, \( \alpha = 1/3 \) is the value at which the fastest growing MHD mode becomes Alfvénic (Pucci & Velli 2014). The condition for that mode to indeed be in the MHD regime is, unsurprisingly, just the reverse of the above, \( S < (L/\rho_s)^2 \). So, in this case (\( \alpha = 1/3 \)), the outcome is particularly simple: if \( S > (L/\rho_s)^2 \) the forming sheet would be disrupted by a semi-collisional mode, and one might expect the reconnection rate to ensue to be \( 0.1 \tau_A^{-1} \) as discussed above; if, instead \( S < (L/\rho_s)^2 \) then the sheet would be disrupted by an MHD mode, and the reconnection rate would presumably be \( Sc^{-1/2} C^{-1} \sim 0.01 \tau_A^{-1} \) (unless a transition to the semi-collisional regime occurs during the plasmoid cascade – see footnote on page 5). For example, in the solar corona, where \( S \sim 10^{13} \) and \( L/\rho_s \sim 10^7 \), we see that the MHD mode would win.

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\footnote{If instead one considers the case of a sinusoidal-like current sheet, (6.1) is replaced by \( \gamma_{max} \tau_A \sim \left( \frac{\rho_s}{L} \right)^{5/8} (15\alpha/8) \geq 1 \); and (6.2) is replaced by \( \left( \frac{\rho_s}{L} \right)^{1/8} S^{−(5\alpha/8)−1/4} < \rho_s/L \). Their intersection now takes place at \( \alpha = 3/10 \). All the same arguments apply to this case.}
Appendix A. Semi-collisional tearing mode scalings in an SP sheet

To confirm the validity of the analytical derivation of Section 2 and the ability of the Viriato code to recover the scalings predicted there, we have performed a set of simulations whose key difference from those reported in the main text lies in the fact that the thickness of initial magnetic profile is now the Sweet-Parker width, $L_{y}S^{-1/2}$, as is assumed throughout Section 2. This initial configuration is not an exact Sweet-Parker sheet because it lacks both the appropriate flows, and the reconnected component of the magnetic field; these are also simplifications adopted in the theoretical derivation, and which can be shown analytically to be justifiable (Loureiro et al. 2007, 2013).

All possible modes are seeded by the introduction of a small amplitude random perturbation at $t = 0$. After an initial transient, we observe the exponential growth of a single mode — the most unstable perturbation. The growth rate and wavenumber (determined by counting the number of arising plasmoids) of this mode are plotted in Fig. 7. Excellent agreement with the theoretical scalings of Eqs. (2.12) and (2.13) is observed.

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