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A rotating and magnetized three-dimensional hot plasma equilibrium in a gravitational field

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Abstract
A rotating and magnetized three-dimensional axisymmetric equilibrium for hot plasma confined by a gravitational field is found. The plasma density and current can exhibit strong equatorial plane localization, resulting in disk equilibria with open magnetic field lines. The associated equatorial plane pinching results in magnetic field flaring, implying a strong gravitational squeezing of the plasma carrying ambient magnetic field lines towards the gravitational source. At high plasma pressure the magnetic field becomes strongly radial outside the disk. The model predicts the rotation frequency bound, the condition for a plasma disk, and the requirement for strong magnetic field flaring.

Analytic axisymmetric three-dimensional confined magnetized plasma equilibria under the combined influence of the gravitational and centrifugal forces are of interest for astrophysical and planetary applications. We present a three-dimensional analytic equilibrium solution for hot, toroidally rotat

Early MHD considerations retaining gravity (Chandrasekhar 1956) assumed strict incompressibility with constant plasma density. Later ideal MHD formulations treating gravity require poloidal flow (Throumoulopoulos & Tasso 2001; McClements & Thyagaraya 2001; Prasanna et al. 1989), which is not permitted kinetically. Also, Throumoulopoulos and Tasso (2001) assume the density is a flux function and considers only dipole solutions, while Prasanna et al. (1989) does not treat the frozen in field flow constraint and tries to satisfy Gauss's law rather than quasineutrality. In addition, these treatments normally retain a toroidal magnetic field that we assume to vanish since it is not required to maintain equilibrium. Work by Lovelace et al. (1986) is noteworthy because it focuses on the thin disk limit with no poloidal flow or toroidal magnetic field. Our magnetic equilibrium model is even simpler. We seek axisymmetric self-similar solutions that lead to an ordinary second order nonlinear differential equation retaining gravity, toroidal rotation, and plasma pressure. We find flared, open magnetic field line
solutions rather than dipole magnetic field lines closing at the localized gravitational source (Throumoulopoulos & Tasso 2001; McClements & Thyagaraya 2001; Prasanna et al. 1989; Ghanbari & Abbassi 2004).

A hot, axisymmetric magnetized plasma rotating toroidally in a gravitational field must satisfy the generalized Grad-Shafranov equation Lovelace et al. (1986). To obtain this equation we assume the axisymmetric magnetic field \( \vec{B} \) is in the poloidal plane by writing

\[
\vec{B} = \nabla \psi \times \nabla \zeta ,
\]

where \( \zeta \) is the toroidal angle and \( \psi \) is the poloidal flux function. We take the velocity \( \vec{V} \) as toroidal

\[
\vec{V} = \Omega R^{2} \nabla \zeta ,
\]

with \( \Omega = c \Phi / d \psi \) the toroidal rotation frequency and \( R \) the cylindrical radius from the axis of symmetry. We assume the magnetic field is frozen into the flow so that \( c \nabla \Phi = \vec{V} \times \vec{B} \) with \( \vec{B} \cdot \nabla \Phi = 0 \), then the electrostatic potential \( \Phi \) must be a flux function to lowest order. Next, the gravitational potential \( G \) is written as

\[
G = -G_{0}M_{0} / r
\]

with \( G_{0} \) the gravitational constant, \( r \) the spherical radius, and \( M_{0} \) the mass of the astrophysical body that is assumed to be a compact source centered at \( r = 0 \), such that \( R = r \sin \theta \) with \( \theta \) the angle from the axis of symmetry. We define our spherical and cylindrical coordinates to satisfy \( r \nabla \theta = R \nabla \zeta \times \nabla r \) and \( \nabla z = R \nabla \zeta \times \nabla r \).

The hot magnetized plasma must adjust its flux surfaces to satisfy Ampere's Law, \( c \nabla \times \vec{B} = 4 \pi \vec{J} \), and total momentum balance under the influence of the attractive force of gravity and the outward centrifugal and plasma pressure forces,

\[
c^{-1} \vec{J} \times \vec{B} = \nabla p + Mn(\nabla \nabla + \vec{V} \cdot \nabla \vec{V}) = Mn(\nabla \nabla - \Omega^{2} R \nabla R) + \nabla (2nT) ,
\]

where \( M \) is the mass of the plasma ions, \( n \) is their density, and the total pressure of the quasineutral plasma \( (Zn = n_{e} = \text{electron density}) \) is \( p = n(T_{i} + ZT_{e}) = 2nT \), with \( T_{i} \) and \( T_{e} \) the ion and electron temperatures and \( Z \) the charge number of the ions. Importantly, the rotation must satisfy total parallel momentum balance

\[
\vec{B} \cdot \nabla [2G - \Omega^{2} R^{2} + (4T/M) / n(\nabla / \eta)] = 0 ,
\]

with \( \vec{B} \cdot \nabla T = 0 = \vec{B} \cdot \nabla \eta \) so both the effective temperature \( T \) and the normalizing density \( \eta \) flux functions. The density and normalizing or "pseudo" density are related via

\[
n = n(\psi, \theta) = \eta(\psi)e^{\kappa(\psi, \theta)} = \eta(\psi)e^{\kappa(\psi, \theta)} ,
\]

to conveniently separate poloidal angle and flux function dependences. The parallel momentum constraint determines \( \kappa \) within a flux function that can be set to zero by absorbing it into the pseudo-density \( \eta \). As a result,

\[
2u^{2} \kappa = \Omega^{2} R^{2} - 2G ,
\]
with \( u^2 = 2T/M \). We see that the flux surfaces and poloidal variation of the density must adjust to maintain parallel force balance between the inward central gravity force and the outward cylindrical rotation force.

The \( \nabla \zeta \) component of total momentum balance requires \( \mathbf{J} \cdot \nabla \psi = 0 \) and radial Ampere's law requires \( \mathbf{J} \cdot \mathbf{r} = 0 \) giving \( \mathbf{J} \cdot \mathbf{B} = 0 \). As a result, there is only toroidal current \( \mathbf{J} = (cMn/B^2) \nabla \psi \cdot (2/Mn) \nabla (nT) + \nabla G - \Omega^2 R V R \nabla \zeta \). (8)

The toroidal component of Ampere's law then gives the desired Grad-Shafranov form
\[
\nabla \cdot (R^{-2} \nabla \psi) = -(4 \pi Mn/R^2 B^2) \nabla \psi \cdot [u^2/R + \nabla u^2 + \nabla \psi + VG - \Omega^2 R V R].
\] (9)

The assumptions used to derive this form satisfy the general constraints for a Maxwellian solution to the drift kinetic or gyrophase averaged Fokker-Planck equation in an axisymmetric system (Hinton & Wong 1985; Catto et al. 1987; Helander 2014). These constraints include \( \nabla \cdot \mathbf{V} = 0 \) and \( \mathbf{B} \cdot \nabla \mathbf{V} \cdot \mathbf{B} = 0 \), as well as \( \mathbf{B} \cdot \nabla \mathbf{T} = 0 \), and are more restrictive than ideal MHD constraints as they do not allow poloidal flow.

We seek a separable solution of the Grad-Shafranov equation by using the technique introduced in Krasheninnikov et al. (1999) for a point dipole magnetic field confining hot plasma in the absence of gravity and rotation. Subsequent work considered hot rotating plasma in the absence of gravity (Catto & Krasheninnikov 1999) and hot, stationary plasma confined by gravity (Krasheninnikov & Catto 1999). Here we consider the more general case of a massive localized gravitational source immersed in a magnetic field confining hot, rotating plasma. Unlike (Krasheninnikov & Catto 1999; Krasheninnikov et al. 2000), we find it is unnecessary to introduce a poloidally varying temperature profile. We restrict our attention to a poloidal flux function of the form
\[
\psi = \psi_o H(\mu)(r_0/r)^\alpha,
\] (10)
where \( \mu = \cos \theta \) and our normalization is \( H(\mu=0) = 1 \). In the vacuum limit \( H = 1 - \mu^2 \), and we recover a homogeneous magnetic field for \( \alpha = -2 \), and the point dipole solution if \( \alpha = 1 \), with \( \psi_o \) a constant reference value of the flux function at the reference location \( r_o \) (reference values are denoted by a subscript "o" and defined in the equatorial plane \( \mu = 0 \)). The reference location is arbitrary because of the self-similarity or scale invariance of the solution form of poloidal flux function.

The magnetic field associated with \( \psi \) is then given by
\[
\mathbf{B} = B_o \left( \frac{r_0}{r} \right)^{\alpha+2} \left[ \frac{\nabla r}{\alpha} \frac{dH}{d\mu} + \frac{H r \nabla \theta}{\sin \theta} \right],
\] (11)
with \( B_o = -\alpha \psi_o / r_0^2 \). For boundary conditions we demand up-down symmetry,
\[
\frac{dH}{d\mu}\bigg|_{\mu=0} = 0,
\] (12)
and require the poloidal magnetic field vanish at the poles.
\[ H(\mu \to \pm 1)/\sqrt{1-\mu^2} \to 0. \]  

(13)

To obtain an ordinary differential equation for \( H \), the expressions \( R/\tau_0 = (\psi_0/H/\psi)^{1/2} (1-\mu^2)^{1/2} \) and \( \tau/\tau_0 = (\psi_0/H/\psi)^{1/2} \) suggest taking the Keplerian form

\[ \Omega^2 = \Omega_0^2 (\psi/\psi_0)^{3/2} \propto 1/\tau^3. \]  

(14)

In addition, to make \( \kappa \) independent of \( \psi \) we take \( u^2 = u_o^2 (\psi/\psi_0)^{1/2} \). For these choices

\[ \kappa = (\Omega_0^2 u_o^2 / 2u_o^2) (1-\mu^2) H^{2/\alpha} + (G_0M_0/\tau_0 u_0^2) H^{1-\alpha}, \]  

(15)

where we also define

\[ \kappa_o = (\Omega_0^2 u_o^2 / 2u_o^2) + (G_0M_0/\tau_0 u_0^2), \]  

(16)

and

\[ n_o = n(\psi = \psi_0, \mu = 0) = \eta_o e^{\kappa_o}. \]  

(17)

Finally, returning to the Grad-Shafranov equation we obtain an ordinary differential equation for \( H \) by taking the density to be

\[ n = \eta e^\kappa = n_o (\eta/\eta_o) e^{\kappa - \kappa_o} = n_o (\psi/\psi_o)^{2+1/\alpha} e^{\kappa - \kappa_o}. \]  

(18)

Inserting this dependence for \( n \) along with our other choices, we are led to define the positive constants

\[ g = \frac{8\pi G_0M_0\eta_n}{\Omega_0^2} = \frac{8\pi G_0M_0\eta_n}{\Omega_0^2} \mid_{\eta = 0}, \]  

(19)

\[ \omega^2 = \frac{4\pi M_0 \Omega_0^2 r_o^2}{B_o} = \frac{4\pi M_0 \Omega_0^2 r_o^2}{B_o} \mid_{\eta = 0}, \]  

(20)

and

\[ \beta = \frac{8\pi M_0 \eta_n^2 u_o^2}{B_o} = \frac{8\pi n (T_i + Z_i)}{B_o} \mid_{\eta = 0} = \frac{8\pi n u_o^2}{B_o} \mid_{\eta = 0}, \]  

(21)

corresponding to the ratios of the gravitational, rotational, and thermal energies, respectively, normalized by the magnetic energy. As indicated, these parameters are constants that do not vary in the equatorial plane. Using them, the Grad-Shafranov equation becomes a nonlinear second order ordinary differential equation for \( H \):

\[ \frac{d^2 H}{d\mu^2} + \frac{1}{2} (\alpha + 1) \frac{H}{1-\mu^2} H = \alpha \left[ \frac{\eta}{g} H^{1-\alpha} - \omega^2 (1-\mu^2) H^{2/\alpha} - (\alpha + 2) \beta \right] H^{1+4/\alpha} e^{\kappa - \kappa_o}. \]  

(22)

A slightly different form for this equation integrated from \( \mu = 0 \) to \( \mu = 1 \) gives a useful integral constraint on \( \alpha \)

\[ (\alpha - 1)(\alpha + 2) \int_0^1 d\mu H = \alpha \int_0^1 d\mu (1-\mu^2) H^{1+4/\alpha} \left[ \frac{\eta}{g} H^{1-\alpha} - \omega^2 (1-\mu^2) H^{2/\alpha} - (\alpha + 2) \beta \right] e^{\kappa - \kappa_o}. \]  

(23)

The preceding equations agree with the various cases considered in (Krasheninnikov et al. 1999, 2000; Catto & Krasheninnikov 1999; Krasheninnikov & Catto 1999).

We notice that there is a possible problem with singular integrals as the poles are
approached due to the g term in $\kappa - \kappa_0$ if $\alpha > 0$ (closed flux surface, dipolar solutions) since $H \rightarrow 0$. Therefore, by retaining gravity we are forced to consider $\alpha < 0$. We first examine the vacuum root $\alpha = -2$ before focusing more generally on $\alpha < 0$. This root corresponds to the limit of a compact object, gravitationally and magnetically confining toroidally rotating hot plasma immersed in a homogeneous magnetic field 

$$\mathbf{B} = B_o [\cos \theta \mathbf{v}_r - \sin \theta \mathbf{v}_\theta] = B_o \mathbf{v}_z.$$ 

The $\alpha = -2$ root is not altered by finite $\beta$ effects in the absence of rotation and gravity.

For $\alpha < 0$ flux surfaces are open to infinity and, in the presence of gravity when $\alpha > -2$, flare out from the localized source at the origin with the flare vanishing as $\alpha \rightarrow -2$. To see this we first observe that for $g = 0$ there is an exact solution $H = (1 - \mu^2)^{\alpha/2}$ of the Grad-Shafranov equation for arbitrary $\beta$, giving an axial field $\mathbf{B} = B_o (r_0/R)^{2\alpha} \mathbf{v}_z$ and

$$\alpha + 2 = -\omega^2/(\beta + 1),$$

(24)

for $\omega^2 \sim 1 \sim \beta$ and $\alpha < 0$. However, $\alpha < -2$ so the magnetic field will increase with radius. This axially or cylindrically symmetric $z$ independent solution is not of interest in the presence of gravity, but it suggests that any departure from pure cylindrical symmetry must be due to gravity and have $\alpha > -2$, the situation we now focus on.

To avoid solutions with magnetic fields increasing with radius we first consider the small $g$ and $\omega^2$ solution for the $\alpha = -2$ root with $H = 1 - \mu^2$. Inserting this solution into the integral constraint, we obtain the $g \sim \omega^2 << 1 \sim \beta$ result

$$2(\alpha - 1)(\alpha + 2)/3 = \alpha \int_0^1 d\mu \left[ \frac{g}{2} \sqrt{1 - \mu^2 - \omega^2 - (\alpha + 2)\beta} \right] e^{-\mu^{1/2} - \mu^{3/2}/2}. \tag{25}$$

For $g << \beta$ only a weak density departure from cylindrical symmetry is allowed giving

$$\alpha = -2 + [(\pi g/8) - \omega^2]/(\beta + 1), \tag{26}$$

where $(\pi g/8) > \omega^2$ is required for a physically interesting solution that keeps $0 > \alpha > -2$ and makes the magnetic field fall off with distance and flare slightly. A more interesting limit is $\beta << g \sim \omega^2 << 1$ which requires strong poloidal density variation and for which the integral constraint reduces to

$$\alpha = -2 + (g - 2\omega^2)(\pi g/8\beta)^{1/2}. \tag{27}$$

This limit is an extension of the $\omega^2 = 0$ result in (Krasheninnikov & Catto 1999; Krasheninnikov et al. 2000) and requires $g > 2\omega^2$ for a physically interesting, slightly flared magnetic field decreasing with radius.

The result for $\beta << g \sim \omega^2 << 1$ has the important feature that the plasma density, and therefore its pressure and the current density, are strongly localized to the equatorial plane since $n/n_o = (\psi/\psi_o)^{2^3/\alpha} e^{-(\psi/\beta)^{(2^{1/2} - 1)/2} \omega^2}$, with a disk thickness $\Delta = R(2\beta/g)^{1/2}$. This plasma disk behavior suggests looking for solutions in the vicinity of the equatorial plane more generally (Lovelace et al. 1986).
To find more general disk solutions allowing the magnetic field to flare while being localized to the equatorial plane we must consider \( g - 2 \omega^2 >> \beta \) so that the exponential dependence \( e^{k \omega^2} \) in the Grad-Shafranov equation provides the desired localization about \( \mu = 0 \). Then we can find plasma disk solutions with strong poloidal variation. To do so we begin by approximating the Grad-Shafranov equation by

\[
d^2H/d\mu^2 = \alpha[(g/2) - \omega^2 - (\alpha + 2)\beta] e^{(\alpha/2-H)/\alpha},
\]

where now both rotation and gravity enter the exponential dependence from the density.

To extend our previous results we note that on the right side of the differential equation for \( H \) we are only interested in the region about \( H = 1 \) so we may use

\[
H^{1/\alpha} = e^{(1/\alpha)\mu H} = e^{(1/\alpha)(H-1)} = 1 + (H-1)/\alpha + ..., \tag{29}
\]

in \( e^{k \omega^2} \) and then assume \( g - 2 \omega^2 >> \beta \) to obtain strong exponential decay away from the equatorial plane. We then obtain

\[
d^2H/d\mu^2 = \alpha [(g/2) - \omega^2 - (\alpha + 2)\beta] e^{(g-2\omega^2)(1-H)/\alpha \beta}. \tag{30}
\]

For \( 0 > \alpha > -2 \) and \( H \leq 1 \), we must assume \( g > 2 \omega^2 \) to obtain flared, decreasing magnetic field solutions localized about the equatorial plane. Solutions strongly localized to the equatorial plane in the presence of gravity are not possible for \( g < 2 \omega^2 \) and \( 0 > \alpha > -2 \) since the rotation is too strong for the plasma to be gravitationally confined.

Multiplying our simplified Grad-Shafranov equation by \( dH/d\mu \) and integrating from \( H = 1 \) (or \( \mu = 0 \)) to \( H < 1 \) (or \( \mu > 0 \)) gives

\[
dH/d\mu = \alpha \sqrt{\beta(g-2\omega^2)[1 - e^{(g-2\omega^2)(1-H)/\beta}]/[g-2\omega^2 - 2(\alpha + 2)\beta]} \,
\]

where we select the negative root to make \( dH/d\mu < 0 \). To find \( H \) we use \( \int dx/\sqrt{1-e^{-x}} = 2 \tanh^{-1} \sqrt{1-e^{-x}} \) to obtain

\[
\frac{g-2\omega^2}{\alpha \beta}(H-1) = x = -\ln[1 - \tanh^2(\sigma \mu/2)] \to \begin{cases} (\sigma \mu/2) + ... & \sigma | \mu |/2 < 1 \\ \pm \sigma \mu - \ln 4 + ... & \sigma | \mu |/2 \geq 1 \end{cases}, \tag{32}
\]

where \( \sigma = \sqrt{(g-2\omega^2)[g-2\omega^2 - 2(\alpha + 2)\beta]/\beta} \) and the upper (lower) sign is for \( \mu > 0 \) (\( \mu < 0 \)). Consequently, for \( x = (g-2\omega^2)(H-1)/\alpha \beta >> 1 \) we obtain a solution strongly localized about the equatorial plane that results in only a small departure from the vacuum solution \( H = 1 - \mu^2 \) that remains an adequate approximation elsewhere. Moreover, as \( \beta \) increases, \( x = (g-2\omega^2)(H-1)/\alpha \beta \geq 1 \) becomes smaller, and the \( H = 1 \) solution extends to \( \mu^2 = 1 \) by satisfying \( d^2H/d\mu^2 = 0 \) since \( x = (g-2\omega^2)(H-1)/\alpha \beta = \pm \sigma \mu \).

Using the preceding approximations for \( x >> 1 \) on the right side of the integral constraint with \( z << 1 \) and the vacuum solution \( H = 1 - \mu^2 \) inserted on the left side yields

\[
\frac{4(\alpha - 1)(\alpha + 2)}{3 \alpha} \approx \sqrt{\frac{\beta[g-2\omega^2 - 2(\alpha + 2)\beta]}{(g-2\omega^2)}} \int_z^1 dz/\sqrt{1-z} = 2 \sqrt{\frac{\beta[g-2\omega^2 - 2(\alpha + 2)\beta]}{(g-2\omega^2)}} \tag{33}
\]
where \( z = e^{(g-2\omega^2)x}/(1-H/\alpha\beta) = e^{-x} \) and we have assumed \( g-2\omega^2 \gg -\alpha\beta \) to make the lower limit zero. For \( \beta \ll 1 \) we recover
\[
\alpha + 2 = \sqrt{\beta},
\]
as found in (Krasheninnikov & Catto 1999) without rotation. In this case we see more generally that gravity and rotation are altering the flux surface shape without explicitly entering the eigenvalue \( \alpha \). We are only able to find a solution if \( 2\omega^2 < g \) so that the gravitational force is strong enough to balance the centrifugal force. The localized equatorial plasma disk solution for \( \beta \ll 1 \) is then \( H - 1 = \mp 2\mu \sqrt{\beta} \), giving
\[
x = \pm (g-2\omega^2)\mu/\sqrt{\beta}
\]
and a disk thickness \( \Delta = R\beta^{1/2}/(g-2\omega^2) \), and requiring \( g-2\omega^2 >> \sqrt{\beta} \).

As \( \beta \) becomes larger, \( x \) becomes smaller so the \( x \geq 1 \) solution extends over the entire plasma. Again assuming \( g-2\omega^2 \gg -\alpha\beta \), we consider \( 0 < -\alpha \ll 2 \) and note that for
\[
H - 1 = \pm \alpha\mu\sqrt{\beta(g-2\omega^2-4\beta)/(g-2\omega^2)}
\]
to vanish at \( \mu = \pm 1 \), we need \( H - 1 = \mp \mu \), giving
\[
0 < -\alpha = (g-2\omega^2)/[(\beta(g-2\omega^2-4\beta)] << 2,
\]
a result that can also be found from the integral constraint, and is consistent with the \( \omega^2 = 0 \) and \( g >> \beta \) result of (Krasheninnikov & Catto 1999). We require \( g-2\omega^2 > 4\beta >> 1 \), and
\[
x = \pm \mu \sqrt{(g-2\omega^2)(g-2\omega^2-4\beta)/\beta}
\]
so the disk width is \( \Delta = R\beta^{1/2}/[(g-2\omega^2)(g-2\omega^2-4\beta)]^{1/2} \).

These results indicate that once \( g-2\omega^2 > 4\beta >> 1 \), the currents associated with gravity, rotation, and diamagnetism become localized to the equatorial disk in such a way as to cause strong flaring in the magnetic field. In particular, when \( \beta \to \infty \) so that \( \alpha \to 0 \), the strongly flared magnetic field becomes nearly radial outside the equatorial plane where it is given by
\[
\vec{B} = B_\theta(r_o/r)^2[(\mp\alpha^{-1}\nabla r - r\sqrt{(1\mp\mu)/(1\pm\mu)}\nabla \theta).}
\]
This expression fails right at the equatorial plane since the \( \sigma |\mu| \geq 1 \) expansion breaks down at very small \( \mu \).

Our \( \beta >> 1 \) limit can also be examined for larger -\( \alpha \) by picking \( \alpha \) such that \( H - 1 = . \pm \mu = \pm \alpha \mu \sqrt{\beta (g-2\omega^2 - 2(\alpha + 2)\beta)/(g-2\omega^2)} \). Then \( \alpha = -1 \) for \( g-2\omega^2 = 2\beta^2/(\beta - 1) \). In this case the magnetic field has significant flaring outside the equatorial plane, but falls off more slowly, as \( 1/r \):
\[
\vec{B} = B_\theta(r_o/r)(\pm \nabla r - r\sqrt{(1\mp\mu)/(1\pm\mu)}\nabla \theta).
\]
Similarly, \( \alpha = -1/2 \) for \( g-2\omega^2 = 3\beta^2/(\beta - 4) \), while \( \alpha = -3/2 \) for \( g-2\omega^2 = \beta^2/(\beta - 4/9) \).

We find a realistic rotating and magnetized three-dimensional axisymmetric equilibria for hot plasma confined by a gravitational field. The plasma density and current exhibit strong localization to the equatorial plane, resulting in plasma disk equilibria with open magnetic field lines. Dipole equilibria are not possible within the model. Our most
important results are for $g - 2\omega^2 \beta \gg 1$, and for $g - 2\omega^2 \gg \sqrt{\beta}$ and $\beta \ll 1$. The small $\beta$ limit only allows small departures from a cylindrically symmetric homogeneous magnetic field in which the plasma is strongly localized to the equatorial plane. In this limit only weakly flared, open flux surfaces are obtained with the magnetic field "pinched in" at the equatorial plane. The pinching in at the equatorial plane that results in the flare of the open field lines is an indication that the plasma on the ambient magnetic field lines is being gravitationally squeezed towards the massive localized gravitational source rather than it acting as the source of the magnetic field. Our large $\beta$ results lead to substantially more flaring of the magnetic field due to the strong localization of the plasma to the equatorial plane, with the magnetic field even becoming strongly radial outside the equatorial disk as $\beta \rightarrow \infty$. This magnetic field is strongest at the poles, but the magnetic energy is much less than the gravitational, rotational, and plasma energies. However, even though the magnetic field is weak, it is needed to allow the strong poloidal density variation that acts to maintain parallel pressure balance as the radial gravitational and cylindrical centrifugal forces become misaligned outside the equatorial plane.

Our open magnetic field model is admittedly simple, but its simplicity is appealing and very likely a virtue, since it makes predictions on the plasma rotation frequency bound ($g > 2\omega^2$ for $\beta \ll 1$ and $g > 2\omega^2 + \beta$ for $\beta \gg 1$), the condition for having a plasma disk in the equatorial plane ($g \gg \sqrt{\beta}$ for $\beta \ll 1$ and $g \gg \beta$ for $\beta \gg 1$), and the requirement for strong magnetic field flaring ($g \gg \beta \gg 1$) that may be possible to check. In our model no equilibria can be found if gravity is too weak ($g < 2\omega^2$ for $\beta \ll 1$ or $g < 2\omega^2 + \beta$ for $\beta \gg 1$) since then the gravitational force is unable to balance the centrifugal force and the plasma pressure.

The Keplerian rotation frequency of our model can only be directly related to the rotation frequency of the localized gravitational source if co-rotation is possible. Then the logical reference radius is the surface of the massive source. However, at small radii reconnection may take place to prevent co-rotation of the magnetic field and a toroidal magnetic field may generated. It is also important to realize that accretion is not considered because it happens on the slower time scales associated with radiative cooling and momentum transport.

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