On the Normal-Mode Frequency Spectrum
of Kinetic Magnetohydrodynamics

J.J. Ramos

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Plasma Science and Fusion Center
Massachusetts Institute of Technology
Cambridge MA 02139, U.S.A.

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Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge MA, U.S.A.

Abstract

This article presents an explicit proof that, in the kinetic magnetohydrodynamics framework, the squared frequencies of normal-mode perturbations about a static equilibrium are real. This proof is based on a quadratic form for the square-integrable normal-mode eigenfunctions and does not rely on demonstrating operator self-adjointness. The analysis is consistent with the quasineutrality condition without involving any subsidiary constraint to enforce it, and does not require the assumption that all particle orbits be periodic. It applies to Maxwellian equilibria, spatially bounded by either a rigid conducting wall or by a plasma-vacuum interface where the density goes continuously to zero.
1. Introduction

An important question in kinetic magnetohydrodynamics (KMHD) linear theory is whether it has the property that its normal-mode perturbations about a static equilibrium have real squared frequency $\omega^2$, hence marginal stability occurs only at $\omega = 0$. A proof that this is actually the case was put forward in the pioneering papers of (Kulsrud 1962a,b). It followed largely a work by Newcomb, that was quoted by Kulsrud as "to be published" but was made available only in privately circulated notes. These early works claim the stronger result that KMHD has a self-adjointness property like ideal-MHD (Bernstein et al. 1957, 1958). However, they are somewhat obscure and their arguments for self-adjointness (as opposed to just having real eigenvalues) may have questionable points. They were invoked occasionally as needed in subsequent work (Antonsen and Lee 1982, VanDam et al. 1982), but their results were taken at face value. Given the fundamental significance of such results, it is surprising that they were never revisited in a more transparent form and, especially, that the review article of (Kulsrud 1983) did not mention the issue of KMHD self-adjointness and that no rendition of Newcomb's notes was ever published. Thereafter, the interest in such a basic issue somehow seems to have waned.

This article presents an independent and more explicit proof of real $\omega^2$ for KMHD linear normal modes. Here, the term "normal-mode" is used according to its strict definition, as a solution of the linearized KMHD equations with the exactly separable time dependence exp($-i\omega t$) and satisfying appropriate boundary conditions, not as the long-time limit of an initial value solution. The analysis follows the KMHD formulation of (Ramos 2014), derived in turn as a special limit of the theory in (Ramos 2010, 2011). In this approach, which is based on drift-kinetic equations in the reference frame of the complete macroscopic fluid velocity, the electric field is eliminated algebraically and the quasineutrality condition is satisfied identically at the exact non-linear level. The dynamical variables of the linearized normal-mode system are the three components of the fluid displacement vector $\xi$ (or the perturbed fluid velocity $u_1 = -i\omega \xi$) and the non-convective parts of the perturbed ion and electron distribution functions that are even with respect to the sign of the particle parallel velocity (denoted by $\hat{f}_{even}$ with $s$ the species index). This system sets a generalized eigenvalue problem
for $\omega^2$ in the space of five-function vectors made of $(\xi, \hat{f}^{even})$. Then, the proof of real $\omega^2$ follows directly from a quadratic form derived for the $(\xi, \hat{f}^{even})$ eigenfunctions. The theorem regarding the comparison of stability conditions with ideal-MHD (Kulsrud 1962b) follows also immediately from this quadratic form. The eigenfunctions the quadratic form is derived for must be square-integrable, so the proof applies rigorously to squared frequencies that belong to the corresponding point spectrum. A stronger result, that would imply also a real continuum spectrum, would entail showing that the operator that has $\omega^2$ for eigenvalue is self-adjoint. The generalized eigenvalue form of the present formulation can be transformed explicitly to a canonical eigenvalue form in the sense of standard spectral theory. However, within the standard meaning of self-adjointness needed for applicability of the spectral theorem, the explicit operator that $\omega^2$ is here the eigenvalue of is shown not to be self-adjoint.

It is pertinent to discuss in some detail the validity conditions for the present proof, comparing them with the conditions assumed in the early Kulsrud-Newcomb work. The proof to be given here assumes that the plasma has only one ion species (of unit charge for simplicity) and that the equilibrium is static, with Maxwellian distribution functions. Besides being preferred physically, the Maxwellian equilibrium assumption follows from the adopted approach of using the near-Maxwellian formulation of Ramos (2010, 2011) as starting point. However, the proof can be easily generalized to other static equilibrium distribution functions that are constant along the magnetic field, isotropic in velocity space and monotonically decreasing with the energy, as assumed by Kulsrud and Newcomb. The proof does not apply to anisotropic-pressure equilibria or to equilibria with flow (on the fast, sonic scale of KMHD). The present proof is completely general with regard the three-dimensional configuration of the equilibrium magnetic field and does not require the condition assumed by Kulsrud and Newcomb that all unperturbed particle orbits be periodic. The periodic orbit assumption is often encountered in the literature but detracts from the generality of the results, making them applicable only to configurations where the magnetic field lines are closed, where all particles are trapped, or where some symmetry makes the passing particle orbits effectively periodic. The present analysis does not have this restriction and is equally valid for non-periodic passing particles on ergodic magnetic field lines. An additional limitation of Kulsrud’s published work is that it rules out the closed magnetic line case, so it ends up necessitating the restriction to a perfectly plugged magnetic mirror.
configuration. Finally, all proofs of real $\omega^2$ apply to spatially confined plasmas. The published work by Kulsrud, as per its more detailed version of (Kulsrud 1962b), assumes ideal wall (i.e. rigid and perfectly conducting) boundary conditions. The result shown in the present work is valid for equilibria whose spatial boundary is set by either an ideal wall or by a plasma-vacuum interface where the density goes continuously to zero. It does not allow a sharp plasma-vacuum interface with pressure discontinuity supported by equilibrium surface currents as in ideal-MHD, but the required smooth boundary condition that the equilibrium density go continuously to zero at the plasma edge is still the more usual situation.

2. KMHD normal mode system

A system for KMHD normal modes, that bypasses the electric field and fulfills automatically the quasineutrality condition, has been derived in (Ramos 2014). It considers linear perturbations about a static Maxwellian equilibrium. Accordingly, the equilibrium distribution function of each species ($s = i, e$) is

$$f_{Ms0} = \left( \frac{m_s}{2\pi} \right)^{3/2} \frac{n_0}{T_{s0}^{3/2}} \exp \left( -\frac{m_s v_r^2}{2T_{s0}} \right), \quad (1)$$

with density and temperatures that are uniform in the direction of the equilibrium magnetic field,

$$\mathbf{B}_0 \cdot \nabla n_0 = 0, \quad \mathbf{B}_0 \cdot \nabla T_{s0} = 0, \quad (2)$$

and satisfy the force balance equation

$$\mathbf{j}_0 \times \mathbf{B}_0 = (\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 = \nabla [n_0(T_{i0} + T_{e0})]. \quad (3)$$

The perturbation variables have the exponential time dependence $\partial / \partial t = -i\omega$ and the fluid displacement vector $\xi = i\mathbf{u}_1 / \omega$ replaces the perturbed single-fluid velocity $\mathbf{u}_1$. Then, the normal-mode equation for $\xi$ is the linearized fluid momentum conservation equation

$$\omega^2 \rho_0 \xi = -\mathbf{j}_0 \times \mathbf{B}_1 - \mathbf{j}_1 \times \mathbf{B}_0 + \sum_{s=i,e} \left\{ \nabla p_{s\perp1} + \nabla \cdot \left[ (p_{s||1} - p_{s\perp1}) \mathbf{b}_0 \mathbf{b}_0 \right] \right\}, \quad (4)$$

where $\rho_0 = (m_i + m_e)n_0$ denotes the equilibrium mass density, $\mathbf{b}_0 = \mathbf{B}_0 / B_0$ denotes the unit vector in the direction of the equilibrium magnetic field, and the perturbed magnetic field and perturbed
electric current are given by
\[ B_1 = \nabla \times (\xi \times B_0) \]  
(5)
and
\[ j_1 = \nabla \times B_1 . \]  
(6)

The perturbed parallel and perpendicular pressures of each species are made of convective plus intrinsically kinetic parts:
\[ p_{s\parallel} = -\xi \cdot \nabla(n_0 T_{s0}) + \hat{p}_{s\parallel} = -\xi \cdot \nabla(n_0 T_{s0}) + \frac{m_s}{2} \int d^3 v' v_\parallel^2 \hat{f}_s \]  
(7)
and
\[ p_{s\perp} = -\xi \cdot \nabla(n_0 T_{s0}) + \hat{p}_{s\perp} = -\xi \cdot \nabla(n_0 T_{s0}) + \frac{m_s}{2} \int d^3 v' v_\perp^2 \hat{f}_s , \]  
(8)
where \( \hat{f}_s \) is the non-convective part of the perturbed distribution function \( f_{s1} \):
\[ \hat{f}_s = f_{s1} + \xi \cdot \frac{\partial f_{Ms0}}{\partial x} . \]  
(9)
The phase-space velocity variable used in this formulation is the random velocity in the reference frame of the macroscopic flow, \( v' = v - u \). Thus, the distribution function moments that determine the pressure tensors in (7,8) do not involve any subtraction of macroscopic velocity components. The theory applies to the zero-Larmor-radius limit of a strongly magnetized collisionless plasma, so all the distribution functions are independent of the gyrophase angle. Only the part of \( \hat{f}_s \) that is even with respect to the sign of \( v'_\parallel \) contributes to the relevant moments in (7,8). Considered as a function \( \hat{f}_s^{even}(v', \lambda, x) \), where \( \lambda = v^2_{\perp}/(v^2 B_0) \) is the ratio of the magnetic moment to the kinetic energy in the moving frame, such even distribution function was shown in (Ramos 2014) to satisfy the normal-mode drift-kinetic equation:
\[ \omega^2 \hat{f}_s^{even} + v^2 (1 - \lambda B_0)^{1/2} b_0 \cdot \frac{\partial}{\partial x} \left[ (1 - \lambda B_0)^{1/2} b_0 \cdot \frac{\partial \hat{f}_s^{even}}{\partial x} \right] = \]
\[ = \left\{ \frac{v^2}{n_0 T_{s0}} (1 - \lambda B_0)^{1/2} b_0 \cdot \frac{\partial}{\partial x} \left[ (1 - \lambda B_0)^{1/2} \hat{f}_s \right] - \frac{\omega_m^2 v^2}{T_{s0}} Q \right\} f_{Ms0} . \]  
(10)
Here and in what follows, the phase-space parallel gradient \( b_0 \cdot \partial/\partial x \) is taken at constant \( v' \) and \( \lambda \).
The coefficient function $F_{s||}(x)$ represents the contribution of the considered species to the parallel magnetofluid force density

$$F_{s||} = b_0 \cdot \nabla \hat{p}_{s||} - \left( \hat{p}_{s||} - \hat{p}_{s\perp} \right) b_0 \cdot \nabla \ln B_0 ,$$

such that the parallel component of (4) is

$$\omega^2 \rho_0 \xi_{||} = \hat{F}_{i\parallel} + \hat{F}_{e\parallel} .$$

(12)

The analysis of (Ramos 2014), or a direct evaluation of (11), shows that $F_{s||}$ can be considered as a functional of $\hat{f}_{\text{even}}$ through the relationship

$$F_{s||} = m_s \int d^3v' v'^2 (1 - \lambda B_0) b_0 \cdot \frac{\partial \hat{f}_{\text{even}}}{\partial x}$$

(13)

where, in terms of the $(v', \lambda)$ variables, the velocity-space volume integral is

$$\int d^3v' \ldots = 2\pi B_0 \int_0^\infty dv' v'^2 \int_0^{1/B_0} d\lambda \left(1 - \lambda B_0\right)^{-1/2} \left(\ldots\right)^{\text{even}} .$$

(14)

The other coefficient function, $Q(\lambda, x)$, couples the drift-kinetic equation (10) to the fluid displacement $\xi$ and is

$$Q(\lambda, x) = \frac{1}{2} \lambda B_0 \nabla \cdot \xi + \left(1 - \frac{3}{2} \lambda B_0\right) b_0 \cdot \left[(b_0 \cdot \nabla)\xi\right]$$

(15)

or, separating the contributions of the parallel and perpendicular components $\xi_{||}$ and $\xi_{\perp}$,

$$Q(\lambda, x) = (1 - \lambda B_0)^{1/2} b_0 \cdot \frac{\partial}{\partial x} \left[(1 - \lambda B_0)^{1/2} \xi_{||}\right] + \frac{1}{2} \lambda B_0 \nabla \cdot \xi_{\perp} - \left(1 - \frac{3}{2} \lambda B_0\right) \xi_{\perp} \cdot \kappa_0$$

(16)

where $\kappa_0 = (b_0 \cdot \nabla) b_0$ is the equilibrium magnetic curvature. Equation (16) will be taken to define $Q$ as a functional of $(\xi_{||}, \xi_{\perp})$.

It is useful to consider separately the parallel and perpendicular components of the momentum conservation equation (4) for the fluid variable $\xi$. As discussed before, the parallel component yields (11,12), which are coupled to the kinetic variables by (13). The perpendicular component can be written in shorthand form as

$$\omega^2 \rho_0 \xi_{\perp} = F_{\perp}^F + \hat{F}_{i\perp} + \hat{F}_{e\perp} ,$$

(17)
where

\[ F^p_{\perp} = -\left\{ j_0 \times \left[ \nabla \times (\xi_{\perp} \times B_0) \right] \right\}_{\perp} - \left\{ \nabla \times \left[ \nabla \times (\xi_{\perp} \times B_0) \right] \right\}_{\perp} \times B_0 - \nabla_{\perp} \left[ \xi_{\perp} \cdot \nabla (n_0 T_{00} + n_0 T_{c0}) \right] \]  

(18)

and

\[ \hat{F}_{s,\perp} = \nabla_{\perp} \hat{p}_{s,\perp} + (\hat{p}_{s,\parallel} - \hat{p}_{s,\perp}) \kappa_0 . \]  

(19)

The term \( F^p_{\perp} \), to be considered as a functional of \( \xi_{\perp} \), is the linearized magnetofluid force density in "perpendicular ideal-MHD", i.e. the ideal-MHD model closed with the pressure evolution equation \( \partial p/\partial t + u \cdot \nabla p = 0 \) that accounts just for convection and ignores the compressional effect. Such compressional effect is properly described by the intrinsically kinetic terms \( \hat{F}_{s,\perp} \), given as functionals of \( \hat{f}^{\text{even}} \) by

\[ \hat{F}_{s,\perp} = \nabla_{\perp} \left[ \frac{m_s}{2} \int d^3v' v'^2 \lambda B_0 \hat{f}^{\text{even}} \right] + \left[ m_s \int d^3v' v'^2 \left( 1 - \frac{3}{2} \lambda B_0 \right) \hat{f}^{\text{even}} \right] \kappa_0 . \]  

(20)

Equations (10,12,17), along with the definitions (13,16,18,20), constitute the present formulation of the closed KMHD normal-mode system. They pose a generalized eigenvalue problem of the form

\[ \omega^2 \mathcal{N}[\xi_{\parallel}, \xi_{\perp}, \hat{f}^{\text{even}}] = \mathcal{M}[\xi_{\parallel}, \xi_{\perp}, \hat{f}^{\text{even}}] , \]

where \( \mathcal{N} \) and \( \mathcal{M} \) are linear operators in the space of five-function vectors made of \( (\xi_{\parallel}, \xi_{\perp}, \hat{f}^{\text{even}}) \). Some unconventional features of this system distinguish it from the traditional formulations, including those of Kulsrud’s and Newcomb’s. Neither the system nor its derivation involve the parallel electric field which was eliminated from the initial, non-linear Vlasov equation by the transformation to the reference frame of the complete macroscopic fluid velocity (Ramos 2011). Then, the system is identically consistent with the quasineutrality condition without the need of any enforcing constraint. Substituting (13) for \( \hat{F}_{s,\parallel} \) and (15) for \( Q \) in the drift-kinetic equation (10), taking its \( \int d^3v' \) moment and carrying out the integral of the term proportional to \( f_{M_{s0}} \), and dividing by \( \omega^2 \) with \( \omega \neq 0 \), yields

\[ \int d^3v' \hat{f}^{\text{even}} = -n_0 \nabla \cdot \xi . \]

(21)

Adding the convective piece,

\[ \int d^3v' f_{s1} = -\xi \cdot \nabla n_0 - n_0 \nabla \cdot \xi . \]

(22)
Therefore, the same fluid continuity expression for the perturbed density is obtained from the moment of the perturbed distribution function of either species and quasineutrality is automatically guaranteed.

3. Quadratic form

A quadratic form can be derived for the square-integrable eigenfunctions of the normal-mode system (10,12,17), which will yield easily the property that their squared frequencies are real and the comparison criterion with ideal-MHD stability (Kulsrud 1962b). Multiplying (10) by $T_s \hat{f}_{\text{even}}^* / f_{M_s0}$ and integrating over velocity-space, one gets the relation

$$
\omega^2 T_s \int d^3 \mathbf{v}' \frac{|\hat{f}_{\text{even}}|^2}{f_{M_s0}} + \omega^2 m_s \int d^3 \mathbf{v}' v'^2 Q \hat{f}_{\text{even}}^* =
$$

$$
= -2\pi B_0 T_s \int_0^\infty dv' \frac{v'^4}{f_{M_s0}} \int_0^{1/B_0} d\lambda \mathbf{b}_0 \cdot \frac{\partial}{\partial \mathbf{x}} \left[ (1 - \lambda B_0)^{1/2} \mathbf{b}_0 \cdot \frac{\partial \hat{f}_{\text{even}}^*}{\partial \mathbf{x}} \right] \hat{f}_{\text{even}}^* +
$$

$$
+ \frac{2\pi B_0}{n_0} \int_0^\infty dv' \frac{v'^4}{f_{M_s0}} \int_0^{1/B_0} d\lambda \mathbf{b}_0 \cdot \frac{\partial}{\partial \mathbf{x}} \left[ (1 - \lambda B_0)^{1/2} \hat{F}_s \right] \hat{f}_{\text{even}}^*. \quad (23)
$$

The parallel derivatives of the right hand side can be transferred to $\hat{f}_{\text{even}}^*$ after completing a total parallel gradient. Recalling also that $v'^2 (1 - \lambda B_0) = v'^2$, this velocity-space-integrated form becomes

$$
\omega^2 T_s \int d^3 \mathbf{v}' \frac{|\hat{f}_{\text{even}}|^2}{f_{M_s0}} + \omega^2 m_s \int d^3 \mathbf{v}' v'^2 Q \hat{f}_{\text{even}}^* =
$$

$$
= T_s \int d^3 \mathbf{v}' \frac{v'^2}{f_{M_s0}} \left| \mathbf{b}_0 \cdot \frac{\partial \hat{f}_{\text{even}}}{\partial \mathbf{x}} \right|^2 - \frac{\hat{F}_s \parallel}{n_0} \int d^3 \mathbf{v}' v'^2 \mathbf{b}_0 \cdot \frac{\partial \hat{f}_{\text{even}}^*}{\partial \mathbf{x}} + \mathbf{B}_0 \cdot \nabla(...) , \quad (24)
$$

where the ellipsis is used to indicate some function of $\mathbf{x}$ whose specifics will not be needed. Substituting now the definition (13) of $\hat{F}_s \parallel$,

$$
\omega^2 T_s \int d^3 \mathbf{v}' \frac{|\hat{f}_{\text{even}}|^2}{f_{M_s0}} + \omega^2 m_s \int d^3 \mathbf{v}' v'^2 Q \hat{f}_{\text{even}}^* =
$$

$$
= T_s \int d^3 \mathbf{v}' \frac{v'^2}{f_{M_s0}} \left| \mathbf{b}_0 \cdot \frac{\partial \hat{f}_{\text{even}}}{\partial \mathbf{x}} \right|^2 - \frac{m_s}{n_0} \left| \int d^3 \mathbf{v}' v'^2 \mathbf{b}_0 \cdot \frac{\partial \hat{f}_{\text{even}}^*}{\partial \mathbf{x}} \right|^2 + \mathbf{B}_0 \cdot \nabla(...) . \quad (25)
$$
The next step is to substitute the definition (16) in the term that contains the function \( Q \),
\[
m_s \int d^3 \mathbf{v'} v'^2 Q \hat{f}^{even}_s * = 2\pi m_s B_0 \int_0^\infty dv' v'^4 \int_0^{1/B_0} d\lambda \mathbf{b_0} \frac{\partial}{\partial \mathbf{x}} \left[ (1 - \lambda B_0)^{1/2} \right] \hat{f}_s^{even} * +
\]
\[
+ \frac{m_s}{2} (\nabla \cdot \xi_\perp) \int d^3 \mathbf{v'} v'^2 \lambda B_0 \hat{f}^{even}_s * - m_s (\xi_\perp \cdot \mathbf{\kappa}_0) \int d^3 \mathbf{v'} v'^2 \left( 1 - \frac{3}{2} \lambda B_0 \right) \hat{f}^{even}_s *
\]
and to complete again total derivatives to obtain
\[
m_s \int d^3 \mathbf{v'} v'^2 Q \hat{f}^{even}_s * = -m_s \xi_\parallel \int d^3 \mathbf{v'} v'^2 (1 - \lambda B_0) \mathbf{b_0} \frac{\partial \hat{f}^{even}_s *}{\partial \mathbf{x}} -
\]
\[
- m_s \xi_\perp \left[ \nabla \left( \frac{1}{2} \int d^3 \mathbf{v'} v'^2 \lambda B_0 \hat{f}^{even}_s * \right) + \mathbf{\kappa}_0 \int d^3 \mathbf{v'} v'^2 \left( 1 - \frac{3}{2} \lambda B_0 \right) \hat{f}^{even}_s * \right] +
\]
\[
+ \nabla \cdot \left[ \left( \frac{m_s}{2} \int d^3 \mathbf{v'} v'^2 \lambda B_0 \hat{f}^{even}_s * \right) \xi_\perp \right] + \mathbf{B_0} \cdot \nabla (...) \quad ,
\]
which, recalling the definitions (8,13,20) of \( \hat{\mathbf{p}}_s \parallel, \hat{\mathbf{F}}_s \parallel \) and \( \hat{\mathbf{F}}_s \perp \), can be expressed as
\[
m_s \int d^3 \mathbf{v'} v'^2 Q \hat{f}^{even}_s * = - \xi_\parallel \hat{\mathbf{F}}_s \parallel - \xi_\perp \cdot \hat{\mathbf{F}}_s \perp + \nabla \cdot (\hat{\mathbf{p}}_s \perp \xi_\perp) + \mathbf{B_0} \cdot \nabla (...) \quad .
\]

Bringing this to (25) and summing over the two species yields
\[
\omega^2 \sum_{s=i,e} T_{s0} \int d^3 \mathbf{v'} \frac{\hat{f}^{even}_s *}{f_{Ms0}} - \omega^2 \left( \xi_\parallel \sum_{s=i,e} \hat{\mathbf{F}}_s \parallel + \xi_\perp \cdot \sum_{s=i,e} \hat{\mathbf{F}}_s \perp \right) =
\]
\[
= \sum_{s=i,e} \left( T_{s0} \int d^3 \mathbf{v'} \frac{v'^2}{f_{Ms0}} \left| \mathbf{b_0} \cdot \frac{\partial \hat{f}^{even}_s *}{\partial \mathbf{x}} \right|^2 - \frac{m_s}{n_0} \left| \int d^3 \mathbf{v'} v'^2 \mathbf{b_0} \cdot \frac{\partial \hat{f}^{even}_s *}{\partial \mathbf{x}} \right|^2 \right) +
\]
\[
+ \nabla \cdot \left[ (...) \mathbf{B_0} - \omega^2 \left( \sum_{s=i,e} \hat{\mathbf{p}}_s \perp \right) \xi_\perp \right] \quad .
\]

The complex conjugates of (12) and (17) are
\[
\sum_{s=i,e} \hat{\mathbf{F}}_s \parallel = \omega^2 p_0 \xi_\parallel \quad .
\]
and
\[ \sum_{s=i,e} \hat{F}_s^* = \omega^2 \rho_0 \xi^*_\perp - F_{\perp}^r[\xi^*_\perp]. \tag{31} \]

Then, the sought after quadratic form is obtained by substituting (30,31) in (29) and carrying out the \( \mathbf{x} \)-space integral over the plasma volume. The final result reads
\[ \omega^2 \left( \int d^3x \ \xi_{\perp} \cdot \mathbf{F}_{\perp}^r[\xi^*_\perp] + \sum_{s=i,e} \int d^3x \ T_{s0} \int d^3v' \left| \frac{\hat{f}_{\text{even}}}{f_{Ms0}} \right|^2 \right) = |\omega|^4 \int d^3x \ \rho_0 \ \dot{\xi} \cdot \dot{\xi}^* + P + S, \tag{32} \]

where
\[ P = \sum_{s=i,e} \int d^3x \left( T_{s0} \int d^3v' \frac{v_{\parallel}^2}{f_{Ms0}} \ b_0 \cdot \frac{\partial \hat{f}_{\text{even}}}{\partial \mathbf{x}} \right)^2 - \frac{m_s}{n_0} \left| \int d^3v' v_{\parallel}^2 b_0 \cdot \frac{\partial \hat{f}_{\text{even}}}{\partial \mathbf{x}} \right|^2 \tag{33} \]
and \( S \) is a surface integral over the plasma spatial boundary, after applying Green’s theorem to the last term of (29),
\[ S = \int dS \cdot \left[ (...) B_0 - \omega^2 \left( \sum_{s=i,e} \hat{p}_{s_\perp}^* \right) \xi_{\perp} \right]. \tag{34} \]

In the quadratic form (32), the term \( \int d^3x \ \xi_{\perp} \cdot \mathbf{F}_{\perp}^r[\xi^*_\perp] \) is real because the ”perpendicular ideal-MHD” operator \( \mathbf{F}_{\perp}^r \) is known to be self-adjoint (Bernstein et al. 1957, 1958). The term \( P \) (33) is manifestly real and it is positive (meant to include the zero case) because of the Cauchy-Schwarz inequality
\[ \left( \int d^3v' v_{\parallel}^2 f_{Ms0} \right) \left( \int d^3v' \frac{v_{\parallel}^2}{f_{Ms0}} \left| b_0 \cdot \frac{\partial \hat{f}_{\text{even}}}{\partial \mathbf{x}} \right|^2 \right) \geq \left| \int d^3v' v_{\parallel}^2 b_0 \cdot \frac{\partial \hat{f}_{\text{even}}}{\partial \mathbf{x}} \right|^2 \tag{35} \]
with
\[ \int d^3v' v_{\parallel}^2 f_{Ms0} = \frac{n_0 T_{s0}}{m_s}. \tag{36} \]
4. Real squared frequency and comparison with ideal-MHD stability

The integrand of the surface term $S$ (34) is quadratic in the perturbation, so the integral needs to be carried only over the unperturbed equilibrium plasma boundary. With boundary conditions that guarantee this surface term to be zero, the right hand side of the quadratic form (32) is real and positive. Then, two important consequences follow. The first is that the KMHD $\omega^2$ is real. The second is the comparison criterion with ideal-MHD stability. It is known that the necessary and sufficient condition for stability in ”perpendicular ideal-MHD” is that $\int d^3x \, \xi_\perp \cdot \mathbf{F}^\perp[\xi_\perp^\ast]$ , as a functional of $\xi_\perp$, be positive definite (Laval et al. 1965). If this is the case, (32) implies that the KMHD $\omega^2$ is positive. So, the statement can be made that, if an equilibrium is stable in ”perpendicular ideal-MHD”, then it is spectrally stable in KMHD. A possible boundary condition that clearly makes $S = 0$ is to assume the plasma boundary to be at a perfectly conducting rigid wall, which implies $dS \cdot \mathbf{B}_0 = 0$ and $dS \cdot \xi_\perp = 0$.

Another boundary condition that ensures $S = 0$, unless $\omega^2$ is already real and positive, is to assume that the plasma is surrounded by a vacuum region and that the equilibrium plasma-vacuum interface is a closed magnetic surface where the density has dropped continuously to zero. In this case one still has $dS \cdot \mathbf{B}_0 = 0$, hence $S = -\omega^2 \sum_{s=1,\epsilon} \int dS \cdot \xi_\perp \, \hat{p}_{s\perp}^\ast$. Besides, the non-convective part of the perturbed distribution function $\hat{f}_s^{\text{even}}$ must satisfy the $n_0 = 0$ limit of its drift-kinetic equation (10) at the boundary. In this limit, the driving term proportional to $Q$ drops out and (10) reduces to

$$\omega^2 \hat{f}_s^{\text{even}} + \nu^2 (1 - \lambda B_0)^{1/2} \mathbf{b}_0 \cdot \frac{\partial}{\partial x} \left[ (1 - \lambda B_0)^{1/2} \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{\text{even}}}{\partial \mathbf{x}} \right] =$$

$$= \frac{\nu^2}{T_s} (1 - \lambda B_0)^{1/2} \mathbf{b}_0 \cdot \frac{\partial}{\partial x} \left[ (1 - \lambda B_0)^{1/2} \hat{F}_s \right] h_{M,s0} ,$$

where $h_{M,s0} = f_{M,s0}/n_0$ is the Maxwellian normalized to $\int d^3\mathbf{v}^\prime \, h_{M,s0} = 1$, independent of $n_0$. Then, following the same reasoning that led to (25),

$$\omega^2 T_s \int d^3\mathbf{v}^\prime \frac{\hat{f}_s^{\text{even}}}{h_{M,s0}} = \mathcal{P}_s + \mathbf{B}_0 \cdot \nabla(\ldots)$$

where

$$\mathcal{P}_s = T_s \int d^3\mathbf{v}^\prime \frac{v_{\parallel}^2}{h_{M,s0}} \left| \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{\text{even}}}{\partial \mathbf{x}} \right|^2 - m_s \left| \int d^3\mathbf{v}^\prime \, v_{\parallel}^2 \mathbf{b}_0 \cdot \frac{\partial \hat{f}_s^{\text{even}}}{\partial \mathbf{x}} \right|^2 \geq 0$$
and, taking the magnetic surface average \( \langle \ldots \rangle \) that annihilates \( \mathbf{B}_0 \cdot \nabla (\ldots) \),

\[
\omega^2 T_s \left( \int d^3 \mathbf{v}' \left| \hat{j}_{s,\text{even}} \right|^2 \right) = \langle \mathcal{P}_s \rangle \geq 0 .
\] (40)

So, unless \( \omega^2 \) is real and positive, one must have \( \hat{j}_{s,\text{even}} = 0 \) at the boundary. Notice that \( \hat{j}_{s,\text{even}} = 0 \) is also consistent with the parallel momentum equation (12,13) when \( n_0 = 0 \). If \( \hat{j}_{s,\text{even}} \) vanishes, so does its moment \( \hat{p}_{s,\perp} \), thus \( \mathcal{S} = 0 \) too.

All the above requires of course that the integrals of the quadratic form (32) exist, namely that the considered normal modes are proper square-integrable eigenfunctions. So, strictly speaking, the proof given that \( \omega^2 \) is real applies when \( \omega^2 \) belongs to the corresponding point spectrum. A real \( \omega^2 \) continuum spectrum would be demonstrated rigorously if it were possible to cast the normal-mode system in the form of a canonical eigenvalue problem (instead of the generalized eigenvalue problem form used so far) and then prove that the operator that has \( \omega^2 \) for eigenvalue is self-adjoint. This possibility will be explored in the next section, but the explicit operator that the present formalism will arrive at is not self-adjoint in the standard sense needed for applicability of the spectral theorem.

5. Lack of self-adjointness

Introducing the shorthand operator notations \( i k_{\parallel} \equiv \mathbf{b}_0 \cdot \partial / \partial \mathbf{x} \) and

\[
G[\eta(\mathbf{x})] \equiv \frac{v_{\parallel}^2}{2} \nabla \cdot \eta - \left( v_{\parallel}^2 - \frac{v_{\perp}^2}{2} \right) \eta \cdot \mathbf{k}_0 ,
\] (41)

the drift-kinetic equation (10) with the definition (16) for \( Q \) can be written as

\[
\omega^2 \left\{ m_s \frac{f_{M \parallel 0}}{T_{s 0}} \left( i v'_{\parallel} k_{\parallel} (v'_{\parallel} \mathbf{F}_{\parallel}) + G[\mathbf{F}_\perp] \right) + \hat{j}_{s,\text{even}} \right\} = v'_{\parallel} k_{\parallel} \left( v'_{\parallel} k_{\parallel} \hat{j}_{s,\text{even}} \right) + i \frac{f_{M \parallel 0}}{n_0 T_{s 0}} v'_{\parallel} k_{\parallel} \left( v'_{\parallel} \mathbf{F}_{\parallel} \right) \] (42)

So, the generalized eigenvalue form of the normal-mode system, \( \omega^2 \mathcal{N}[\mathbf{k}_{\parallel}, \mathbf{k}_\perp, \hat{j}_{s,\text{even}}] = \mathcal{M}[\mathbf{k}_{\parallel}, \mathbf{k}_\perp, \hat{j}_{s,\text{even}}, \hat{j}_{s,\text{even}}] \), is given by (12,17,42) with the operator \( \mathcal{N} \) defined by their left hand sides. This operator can be easily
inverted by substituting (12) and (17) for $\omega^2 \xi_{||}$ and $\omega^2 \xi_{\perp}$ in the left hand side of (42). The result is
\[
\omega^2 \hat{f}_{s}^{\text{even}} = -m_s \frac{f_{MS0}}{\rho_0 T_{s0}} G \left[ F_{s ||}^{F} \xi_{||} + \sum_{s' = i, e} \hat{F}_{s' ||}^{s ||} \hat{f}_{s' ||} \right] + v'_{||} k_{||} \left( v'_{||} k_{||} \hat{f}_{s} \right) + \\
+i \frac{f_{MS0}}{\rho_0 T_{s0}} v'_{||} k_{||} \left[ v'_{||} \left( m_s \hat{F}_{s ||} \left[ \hat{f}_{s ||} \right] - m_s \hat{F}_{s' ||} \left[ \hat{f}_{s' ||} \right] \right)_{s' \neq s} \right],
\]
(43)
such that (12,17) divided by $\rho_0$ and (43) express the normal-mode system as the canonical eigenvalue problem $\omega^2 (\xi_{||}, \xi_{\perp}, \hat{f}_{s}^{\text{even}}) = \mathcal{N}^{-1} \mathcal{M} [\xi_{||}, \xi_{\perp}, \hat{f}_{s}^{\text{even}}]$. In this way, $\xi_{||}$ is decoupled from the equations (17,43) that form a closed system for $(\xi_{\perp}, \hat{f}_{s}^{\text{even}})$, hence $\xi_{||}$ is determined explicitly by (12) after the $(\xi_{\perp}, \hat{f}_{s}^{\text{even}})$ solution has been obtained. Now, the system of (17) divided by $\rho_0$ and (43) define a canonical eigenvalue problem, $\omega^2 (\xi_{\perp}, \hat{f}_{s}^{\text{even}}) = \mathcal{L} [\xi_{\perp}, \hat{f}_{s}^{\text{even}}]$, in the space of four-function vectors $(\xi_{\perp}, \hat{f}_{s}^{\text{even}})$. This operator can be split into two pieces, $\mathcal{L} = \mathcal{L}^{D} + \mathcal{L}^{C}$, where $\mathcal{L}^{D}$ is the block-diagonal operator
\[
\mathcal{L}^{D} \left[ \xi_{\perp}, \hat{f}_{s}^{\text{even}} \right] = \left( \rho_0^{-1} F_{s}^{F} \xi_{||}, H_s \left[ \hat{f}_{s}^{\text{even}} \right] \right)
\]
(44)
with
\[
H_s \left[ \hat{f}_{s'}^{\text{even}} \right] = v'_{||} k_{||} \left( v'_{||} k_{||} \hat{f}_{s'}^{\text{even}} \right) + \frac{i f_{MS0}}{\rho_0 T_{s0}} v'_{||} k_{||} \left[ v'_{||} \left( m_s \hat{F}_{s ||} \left[ \hat{f}_{s ||} \right] - m_s \hat{F}_{s' ||} \left[ \hat{f}_{s' ||} \right] \right)_{s' \neq s} \right],
\]
(45)
and $\mathcal{L}^{C}$ is the remainder
\[
\mathcal{L}^{C} \left[ \xi_{\perp}, \hat{f}_{s}^{\text{even}} \right] = \left( \rho_0^{-1} \sum_{s' = i, e} \hat{F}_{s' ||}^{s ||} \hat{f}_{s' ||}^{\text{even}} \right), -m_s \frac{f_{MS0}}{\rho_0 T_{s0}} G \left[ F_{s ||}^{F} \xi_{||} + \sum_{s' = i, e} \hat{F}_{s' ||}^{s ||} \hat{f}_{s' ||}^{\text{even}} \right].
\]
(46)

The two blocks of $\mathcal{L}^{D}$ are self-adjoint in their respective subspaces. As is well known (Bernstein et al. 1957, 1958), $\rho_0^{-1} F_{s}^{F}$ is self-adjoint in the space of perpendicular vectors $\xi_{\perp}(\mathbf{x})$ with the scalar product
\[
\langle \eta_{\perp} | \xi_{\perp} \rangle = \int d^3 \mathbf{x} \rho_0 \eta_{\perp}^{*} \cdot \xi_{\perp}.
\]
(47)
The operator $H_s$ is also self-adjoint in the space of the two species distribution functions $f_{s}(v', \lambda, \mathbf{x})$ with the scalar product
\[
\langle (g_s)(f_s) \rangle = \sum_{s = i, e} \int d^3 \mathbf{x} \int d^3 \mathbf{v}' \frac{T_{s0}}{f_{MS0}} g_{s}^{*} f_{s},
\]
(48)
as can be shown by forming the scalar product $\langle (g_s)(H_s[f_{s'}]) \rangle$ and carrying out partial integrations analogous to those in section 3. Assuming the boundary condition $dS \cdot B_0 = 0$, which makes surface terms vanish, the following symmetric expression is obtained

$$\langle (g_s)(H_s[f_{s'}]) \rangle = \sum_{s=i,e} \int d^3x \int d^3\nu' \frac{T_{s0}}{f_{M,s0}} v_{s}^{2} (ik \parallel g_{s}^{*}) (ik \parallel f_{s}) - \int d^3x \frac{m_i m_e}{\rho_0} \left( \frac{1}{m_i} \hat{F}_{i\parallel}[g_{s}^{*}] - \frac{1}{m_e} \hat{F}_{e\parallel}[g_{s}^{*}] \right) \left( \frac{1}{m_i} \hat{F}_{i\parallel}[f_{s}] - \frac{1}{m_e} \hat{F}_{e\parallel}[f_{s}] \right), \quad (49)$$

hence $\langle (g_s)(H_s[f_{s'}]) \rangle = \langle (H_s[g_s])(f_{s}) \rangle$. This means that $\mathcal{L}^D$ is self-adjoint in the complete space of four-function vectors $(\xi_{\perp}, f_{s})$ with a scalar product that combines the subspace definitions (47,48). There is a difficulty here though, because (47) and (48) do not have the same dimensions. Their ratio has the dimension of a squared frequency, which leaves one with two possible options.

The first option is to introduce some real and positive constant $\Omega_0^2$ in the definition of the scalar product,

$$\langle (\eta_{\perp}, g_{s})(\xi_{\perp}, f_{s}) \rangle = \Omega_0^2 \int d^3x \rho_0 \eta_{\perp} \cdot \xi_{\perp} + \sum_{s=i,e} \int d^3x \int d^3\nu' \frac{T_{s0}}{f_{M,s0}} g_{s}^{*} f_{s}, \quad (50)$$

with which one can now evaluate $\langle (\eta_{\perp}, g_{s})|\mathcal{L}^C[\xi_{\perp}, f_{s}] \rangle$. After carrying out partial integrations analogous to those in section 3 and assuming the boundary conditions $dS \cdot B_0 = 0$ and either $dS \cdot \eta_{\perp} = dS \cdot \eta_{\perp} = 0$ or $f_{s} = g_{s} = 0$ at the plasma spatial boundary to make surface terms vanish, the following is obtained:

$$\langle (\eta_{\perp}, g_{s})|\mathcal{L}^C[\xi_{\perp}, f_{s}] \rangle = \sum_{s=i,e} \int d^3x \Omega_0^2 \eta_{\perp} \cdot \hat{F}_{s\perp}[f_{s}] + \frac{1}{\rho_0} \left( \mathcal{F}_{\perp}[\xi_{\perp}] + \sum_{s'=i,e} \mathcal{F}_{s'\perp}[f_{s'}] \right) \cdot \mathcal{F}_{s\perp}[g_{s}^{*}] \right). \quad (51)$$

Clearly, this expression does not have the general symmetry needed to prove that $\mathcal{L}^C$ is self-adjoint. Such symmetry would be manifest if one of the two arguments of the bilinear form, for instance $(\xi_{\perp}, f_{s})$, satisfied

$$\frac{1}{\rho_0} \left( \mathcal{F}_{\perp}[\xi_{\perp}] + \sum_{s'=i,e} \mathcal{F}_{s'\perp}[f_{s'}] \right) = \Omega_0^2 \xi_{\perp} \quad (52)$$

but this is not true in general. Equation (52) holds if $(\xi_{\perp}, f_{s})$ is a normal-mode solution with a positive eigenvalue $\omega^2$ and one sets $\Omega_0^2 = \omega^2$, but this is not sufficient to prove self-adjointness of $\mathcal{L}^C$. 

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Considering (52) as a constraint would be unphysically restrictive because a specific choice would have to be made for \( \Omega_0^2 \) and that would exclude the physical normal modes (except possibly a single one that would have to have a positive \( \omega^2 \)).

The second option is to use the perpendicular velocity \( \mathbf{u}_{1\perp} \), instead of the perpendicular displacement \( \mathbf{\xi}_{1\perp} \), as the primary fluid perturbation variable to form the linearized state vectors \( (\mathbf{u}_{1\perp}, \hat{f}_{s}^{\text{even}}) \). For these, one has the intrinsic scalar product definition free of arbitrary constants

\[
\langle (s_{\perp}, g_{s}) | (\mathbf{u}_{1\perp}, f_{s}) \rangle = \int d^3x \, \rho_0 \, s_{\perp} \cdot \mathbf{u}_{1\perp} + \sum_{s = i, e} \int d^3x \int d^3\mathbf{v} \, \frac{T_{s 0}}{f_{s 0}} \, g_{s}^{*} \, f_{s} .
\]  

(53)

The linear normal-mode system becomes now \( \omega^2 (\mathbf{u}_{1\perp}, \hat{f}_{s}^{\text{even}}) = \mathcal{L}_{\omega} [\mathbf{u}_{1\perp}, \hat{f}_{s}^{\text{even}}] \) where \( \mathcal{L}_{\omega} = \mathcal{L}^D + \mathcal{L}_{\omega}^C \), \( \mathcal{L}^D \) is the same block-diagonal operator defined in (44) and

\[
\mathcal{L}_{\omega}^C [\mathbf{u}_{1\perp}, \hat{f}_{s}^{\text{even}}] = \left( -\frac{i \omega}{\rho_0} \sum_{s' = i, e} \hat{F}_{s'\perp} [\hat{f}_{s'}^{\text{even}}] , -m_{s} \frac{T_{s 0}}{\rho_0 T_{s 0}} G \left[ \frac{i \omega}{\omega} \hat{F}_{s\perp}^{F} [\mathbf{u}_{1\perp}] + \sum_{s' = i, e} \hat{F}_{s'\perp} [\hat{f}_{s'}^{\text{even}}] \right] \right) .
\]  

(54)

As before, \( \mathcal{L}^D \) is independent of \( \omega \) and self-adjoint. However, \( \mathcal{L}_{\omega}^C \) is now an operator that depends on \( \omega \), hence the linear normal-mode system is no longer expressible as a canonical eigenvalue problem and the standard results of spectral theory cannot be applied. Besides, a calculation analogous to the one that led to (51) yields

\[
\langle (s_{\perp}, g_{s}) | \mathcal{L}_{\omega}^C [\mathbf{u}_{1\perp}, f_{s}] \rangle = \sum_{s = i, e} \int d^3x \left\{ -i \omega \, s_{\perp}^{*} \cdot \hat{F}_{s\perp} [f_{s}] + \frac{1}{\rho_0} \left( \frac{i \omega}{\omega} \hat{F}_{s\perp}^{F} [\mathbf{u}_{1\perp}] + \sum_{s' = i, e} \hat{F}_{s'\perp} [f_{s'}^{s}] \right) \cdot \hat{F}_{s\perp} [g_{s}^{s}] \right\} ,
\]  

(55)

which again does not have the symmetry needed for \( \omega \)-dependent self-adjointness. Such symmetry would be exhibited if one of the two arguments of the bilinear form, for instance \( (\mathbf{u}_{1\perp}, f_{s}) \), satisfied

\[
\frac{1}{\rho_0} \left( \frac{i \omega}{\omega} \hat{F}_{s\perp}^{F} [\mathbf{u}_{1\perp}] + \sum_{s' = i, e} \hat{F}_{s'\perp} [f_{s'}^{s}] \right) = i \omega^{*} \mathbf{u}_{1\perp} .
\]  

(56)

Again, this relationship holds if \( (\mathbf{u}_{1\perp}, f_{s}) \) is a normal-mode solution with a positive eigenvalue \( \omega^2 \), but not in general. Besides it being \( \omega \)-dependent, imposing it as a constraint would be unphysically restrictive because that would exclude the physical, unstable normal modes with \( \omega^2 < 0 \).
Even if $L_\omega$ is not self-adjoint, one can still use the relationship

$$\omega^2((\mathbf{u}_{1\perp}, \hat{\mathbf{f}}_{\text{even}})|((\mathbf{u}_{1\perp}, \hat{\mathbf{f}}_{\text{even}})) = \langle (\mathbf{u}_{1\perp}, \hat{\mathbf{f}}_{\text{even}})|L_\omega[\mathbf{u}_{1\perp}, \hat{\mathbf{f}}_{\text{even}}]\rangle$$

(57)

as a quadratic form for any given square-integrable normal-mode $(\mathbf{u}_{1\perp}, \hat{\mathbf{f}}_{\text{even}})$ with eigenvalue $\omega^2$. Always with the assumed boundary conditions that make the surface terms equal to zero, this yields

$$\omega^2 \left( \int d^3\mathbf{x} \rho_0 \mathbf{u}_{1\perp}^* \cdot \mathbf{u}_{1\perp} + \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{v}' \frac{T_{s0}}{f_{Ms0}} |\hat{\mathbf{f}}_{\text{even}}|^2 \right) = \int d^3\mathbf{x} \mathbf{u}_{1\perp}^* \cdot \mathbf{F}_s \left[ \mathbf{u}_{1\perp} \right] +$$

$$+ \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{v}' \frac{T_{s0}}{f_{Ms0}} v'^2 \left| i k || \hat{\mathbf{f}}_{\text{even}} \right|^2 - \int d^3\mathbf{x} \frac{m_i m_e}{\rho_0} \left| \frac{1}{m_i} \hat{F}_i || \hat{f}_{\text{even}} \right|^2 - \frac{1}{m_e} \hat{F}_e || \hat{f}_{\text{even}} \right|^2 +$$

$$+ \sum_{s=i,e} \int d^3\mathbf{x} \left( -i \omega \mathbf{u}_{1\perp}^* \cdot \hat{\mathbf{F}}_s \left[ \hat{\mathbf{f}}_{\text{even}} \right] + i \omega \mathbf{u}_{1\perp} \cdot \hat{\mathbf{F}}_s \left[ \hat{\mathbf{f}}_{\text{even}}^* \right] \right).$$

(58)

Now, setting $\mathbf{u}_{1\perp} = -i \omega \mathbf{e}_\perp$ and substituting (17) and its complex conjugate (31) for $\sum_{s=i,e} \hat{\mathbf{F}}_s || \hat{f}_{\text{even}}$ and $\sum_{s=i,e} \hat{\mathbf{F}}_s || \hat{f}_{\text{even}}^*$ respectively, this quadratic form becomes

$$\omega^2 \left( \int d^3\mathbf{x} \mathbf{e}_\perp \cdot \mathbf{F}_s \left[ \mathbf{e}_\perp \right] + \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{v}' \frac{T_{s0}}{f_{Ms0}} |\hat{\mathbf{f}}_{\text{even}}|^2 \right) = |\omega|^4 \int d^3\mathbf{x} \rho_0 \mathbf{e}_\perp \cdot \mathbf{e}_\perp^* +$$

$$+ \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{v}' \frac{T_{s0}}{f_{Ms0}} v'^2 \left| i k || \hat{\mathbf{f}}_{\text{even}} \right|^2 - \int d^3\mathbf{x} \frac{m_i m_e}{\rho_0} \left| \frac{1}{m_i} \hat{F}_i || \hat{f}_{\text{even}} \right|^2 - \frac{1}{m_e} \hat{F}_e || \hat{f}_{\text{even}} \right|^2.$$

(59)

This can be verified to be equivalent to the initially derived quadratic form (32,33), after recalling the definition (13) of $\hat{F}_s || \hat{f}_{\text{even}}$ and that the product of (12) times its complex conjugate implies

$$|\omega|^4 \int d^3\mathbf{x} \rho_0 \mathbf{e}_\perp \cdot \mathbf{e}_\perp^* = \int d^3\mathbf{x} \frac{1}{\rho_0} \left| \hat{F}_i || \hat{f}_{\text{even}} \right|^2 + \hat{F}_e || \hat{f}_{\text{even}} \right|^2.$$

(60)

6. Parallel electric field

One of the strengths of the formulation advocated in this work is the fact that the electric field was eliminated exactly from the primary Vlasov equation before any asymptotic approximations were
made. All the subsequent kinetic analysis, in particular the KMHD analysis that has been the subject of this article, is carried out without any reference to the electric field which has nevertheless been taken into account rigorously. This avoids the delicate issue of the treatment of its small but necessary parallel component, that the traditional formulations of Larmor-radius-expanded kinetic theory must contend with. In any case, the parallel electric field can be inferred from the solution of the presently formulated KMHD system. In the collisionless, single-fluid KMHD limit, the parallel components of the linearized electron and ion momentum equations yield

\[
E_\parallel = \frac{1}{e n_0} \left( \omega^2 m_e n_0 \xi_\parallel - \hat{F}_e||[\hat{f}_e^{\text{even}}] \right) = -\frac{1}{e n_0} \left( \omega^2 m_i n_0 \xi_\parallel - \hat{F}_i||[\hat{f}_i^{\text{even}}] \right) .
\] (61)

The same value of \( E_\parallel \) is obtained from the electron or ion expressions in (61) because \( \xi_\parallel \) satisfies the total momentum conservation equation (12), which is the linear combination of these two equations that has been used so far. Another independent linear combination eliminates \( \xi_\parallel \) and gives the parallel electric field as the functional of only \( \hat{f}_s^{\text{even}} \),

\[
E_\parallel = \frac{1}{e \rho_0} \left( m_e \hat{F}_e||[\hat{f}_e^{\text{even}}] - m_i \hat{F}_i||[\hat{f}_i^{\text{even}}] \right) .
\] (62)

In terms of this \( E_\parallel \), the normal-mode drift-kinetic equation (43) can be written as

\[
\omega^2 \hat{f}_s^{\text{even}} = -m_s \frac{f_{Ms0}}{\rho_0 \Gamma_{s0}} G \left[ F_\perp[\xi_\perp] + \sum_{s'=i,e} \hat{F}_{s'||}[\hat{f}_{s'}^{\text{even}}] \right] + v_{\|}' k_\parallel \left( v_{\|}' k_\parallel \hat{f}_s^{\text{even}} \right) + i \frac{f_{Ms0}}{\Gamma_{s0}} v_{\|}' k_\parallel \left( v_{\|}' e_s E_\parallel \right)
\] (63)

with \( e_i = -e_e = e \), and the quadratic form (59) can be written as

\[
\omega^2 \left( \int d^3\mathbf{x} \xi_\perp \cdot F_\perp[\xi_\perp] + \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{v}' \frac{T_{s0}}{f_{Ms0}} \left| \hat{f}_s^{\text{even}} \right|^2 \right) = \left| \omega \right|^2 \int d^3\mathbf{x} \rho_0 \xi_\perp \cdot \xi_\perp^* + \sum_{s=i,e} \int d^3\mathbf{x} \int d^3\mathbf{v}' \frac{T_{s0}}{f_{Ms0}} v_{\|}'^2 \left| b_0 \cdot \frac{\partial \hat{f}_s^{\text{even}}}{\partial \mathbf{x}} \right|^2 - \int d^3\mathbf{x} \frac{e^2 \rho_0}{m_i m_e} \left| E_\parallel \right|^2 .
\] (64)

The system of (17,62,63) has the structure of a closed system for \( \xi_\perp, E_\parallel \) and the two distribution functions, like in the traditional formulations such as Kulsrud’s and Newcomb’s. However, \( E_\parallel \) is now given explicitly by (62), hence it can be eliminated outright, instead of being specified implicitly by
the condition that quasineutrality imposes on some electromagnetic potential. Another significant difference is that the present forms of the drift-kinetic equation in the reference frame of the complete fluid velocity (10,43,63) use as phase-space coordinates with good adiabatic invariance properties the magnetic moment and the particle kinetic energy (or combinations thereof). This is in contrast with mainstream kinetic formulations that use as coordinates the magnetic moment and the particle total energy, including its electric potential energy, and have therefore the complication of involving one of the dynamical unknowns (the electric potential) in the definition of the coordinate system and its associated Jacobian.

7. Summary

Some properties of the spectrum of KMHD normal-mode frequencies have been studied with an intrinsically quasineutral formalism that differs from the traditional approaches. This analysis has yielded a novel expression of the energy integral, shown as three equivalent versions of a quadratic form for the square-integrable normal-mode eigenfunctions of the linearized KMHD system about a static equilibrium. The first version (32,33) depends on the three components of the fluid displacement vector $\xi$ and the electron and ion distribution functions, after eliminating the parallel electric field; the second one (59) depends on the two perpendicular components of the fluid displacement and the two distribution functions, after eliminating both the the parallel electric field and the parallel fluid displacement $\xi_\parallel$; the third one (64) depends on the two perpendicular components of the fluid displacement, the parallel electric field and the two distribution functions, after eliminating $\xi_\parallel$. These three expressions of the quadratic form show that the square-integrable normal-mode eigenfunctions about a static equilibrium have real $\omega^2$ eigenvalues. Accordingly, instabilities are purely growing and marginal stability occurs at zero frequency. The first expression (32,33), involving the three components of $\xi$, makes also manifest the comparison criterion with ideal-MHD stability.

The explicit operator that $\omega^2$ is the eigenvalue of in the present formulation is shown not to be self-adjoint. Therefore, even though its point spectrum its shown to be real, no mathematically rigor-
ous statement can be made here about the nature of its continuum spectrum or the possible existence of a complete basis of generalized eigenfunctions. This result is weaker than the one claimed by (Kulsrud 1962a,b), namely a self-adjointness property of KMHD akin to ideal-MHD. On the other hand, the present results are not limited by the restrictive geometrical assumptions made by (Kulsrud 1962a,b). The present results apply to general magnetic confinement configurations, including those with non-periodic passing particles, and do not necessitate a separate consideration of the case of closed magnetic line loops. In addition, they allow for both the ideal wall boundary condition and the more realistic boundary condition of a confined plasma surrounded by a vacuum, provided the plasma-vacuum interface is a closed magnetic surface where the equilibrium density has dropped continuously to zero.

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